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Abstract

Full Text

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POINCARÉ' S THEOREM ON THE UPPER LIMIT OF THE ANGULAR VELOCITY OF ROTATION FOR STELLAR SYSTEMS

(Presented by Academician V. A. Ambartsumian on 26 IV 1957)

In 1885 H. Poincaré proved ⁽¹⁾ that for a liquid mass in equilibrium, rotating as a rigid body, the inequality

$$\omega^2 < 2\pi G\bar{\rho}, \quad (1)$$

must hold, where ω is the angular velocity; G is the gravitational constant; $\bar{\rho}$ is the mean density of the liquid. In the proof of this theorem the outer surface of the liquid mass is essentially used, and therefore the proof is not applicable to discrete systems of gravitating points, such as, for example, all stellar systems. In this connection it is of interest to give another proof, applicable to discrete systems. This note is devoted to this.

On the basis of the virial theorem ⁽²⁾, for any equilibrium discrete system

$$2T + W = 0, \quad (2)$$

where $T = \frac{1}{2} \sum_i m_i v_i^2$ is the kinetic energy; $W = -G \sum_{i>j} \sum_j \frac{m_i m_j}{r_{ij}}$ is the potential energy of the system.

For a system rotating as a rigid body one may write

$$2T = M\hat{v}^2 + J\omega^2, \quad (3)$$

where M is the mass of the system; \hat{v} is the mean-square velocity of the particles relative to coordinates rotating together with the given material system; J is the moment of inertia of the system with respect to the axis of rotation.

Substituting (2) into (3) and discarding the positive quantity $M\hat{v}^2$, we obtain the inequality

$$\omega^2 < -W/J, \quad (4)$$

which represents an analogue of Poincaré's theorem for discrete gravitating systems. It can be given a somewhat more concrete form by noting that $J = Ms^2$, where s is the radius of inertia.

The expression for the potential energy can be simplified by assuming that the masses of the stars are equal, i.e.

$$m_i = m, \quad M = mN, \quad (5)$$

where N is the total number of stars in the system. Then

$$W = -Gm^2 \sum_{i>j} \sum_j \frac{1}{r_{ij}} = -G \frac{m^2 N(N-1)}{2a} \simeq -\frac{GM^2}{2a}, \quad (6)$$

where $\frac{1}{a} = \overline{\left(\frac{1}{r_{ij}}\right)}$ is the mean value of the reciprocals of the relative distances of the stars. In the last term, unity has been neglected in comparison with N . Substituting the expressions for W and J into (4), we obtain:

$$\omega^2 < \frac{GM}{as^2}. \quad (7)$$

For systems of a special form, (4) can be refined, while at the same time giving it a more transparent form. For a homogeneous ellipsoid with semiaxes a, b, c ,

$$W = -\frac{8}{15}\pi^2 G\rho^2 a^2 b^2 c^2 I, \quad (8)$$

where ρ is the density; I is the well-known elliptic integral ⁽³⁾:

$$I = \int_0^\infty \frac{ds}{\sqrt{(a^2+s)(b^2+s)(c^2+s)}}. \quad (9)$$

Directing the z -axis along the axis of rotation, we obtain for the moment of inertia

$$J = \frac{4}{15}\pi\rho abc(a^2 + b^2), \quad (10)$$

and then our inequality takes the form

$$\omega^2 < 2\pi G\rho \frac{abc}{a^2 + b^2} I. \quad (11)$$

In the case of an ellipsoid of revolution ($a = b > c$) we obtain

$$\omega^2 < 2\pi G\rho \frac{a^2 c}{a^2 + b^2} 2 \arccos \frac{c}{a}, \quad (12)$$

which, for a strongly flattened ellipsoid ($c \ll a$), gives approximately

$$\omega^2 < 2\pi G\rho \frac{\pi c}{a}. \quad (13)$$

Finally, for a sphere ($a = b = c$) we obtain

$$\omega^2 < 2\pi G\rho, \quad (14)$$

which agrees exactly with Poincaré's formula (1).

In the general case, for ellipsoidal figures of equilibrium, Poincaré's condition can be written in the form

$$\omega^2 < 2\pi G\rho K, \quad (15)$$

where $K \leq 1$ depends only on the shape of the ellipsoid. The equality sign is in fact not attained, since it corresponds to the case of a sphere, for which, by dynamical considerations, one must have $\omega = 0$.

If we take our Galaxy as an example, for which one may adopt $c/a = 0.1$, then, taking the stellar density (the number of stars in 1 cm^3) in the vicinity of the Sun as $\nu = 3.41 \cdot 10^{-57}$ and the mean mass of one star as equal to one-half a solar mass: $m = 10^{33} \text{ g}$, we obtain for the limit of the angular velocity the value $0''.0044$ per year, which is less than the observed angular velocity of rotation of the Galaxy in the vicinity of the Sun: $\omega = 0''.0068$ ⁽⁴⁾. For equilibrium it is necessary that the stellar density be at least 2.4 times greater than the observed value. This result may be regarded as a confirmation of the well-known fact that our Sun is located outside the principal dynamically stable part of the Galaxy.

It is important to note that Poincaré's inequality must be satisfied the more strongly, the larger the omitted term $M\bar{v}^2$ is in comparison with $J\omega^2$, i.e., the greater the energy of the irregular (residual) motions of the stars is in comparison with the energy of the regular (rotational) motion.

As is known, in stellar systems of the type of normal spirals, to which our Galaxy also belongs, precisely the opposite occurs. Therefore, for such systems the angular velocity cannot differ significantly from the Poincaré limit. For these systems Poincaré's inequality may be replaced by an approximate equality, which will make it possible to find approximately the mean stellar density if the angular velocity is known, and conversely.

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- ⁴ P. P. Parenago, *Course of Stellar Astronomy*, 1954, p. 158.

Note: Figure translations are in progress. See original paper for figures.

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