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Abstract

Full Text

MATHEMATICS

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SOME BOUNDARY-VALUE PROBLEMS FOR EQUATIONS OF MIXED TYPE IN A STRIP AND A HALF-PLANE

(Presented by Academician S. L. Sobolev on 22 IX 1956)

A second-order partial differential equation

$$y^m a(y) \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \left\{ -\frac{n}{y} + b(y) \right\} \frac{\partial U}{\partial y} + c(y)U = 0, \quad (1)$$

where $a(y)$, $b(y)$, $c(y)$ are analytic functions of y and $a(y) > 0$, $m = n = 2k - 1$, or $m = -1$, $n = k - 1$ ($k = 1, 2, 3, \dots$), is an equation of mixed elliptic-hyperbolic type in a domain containing the axis Ox .

Let us consider the following boundary-value problems for equation (1) in the strip

$$-\infty < x < +\infty, \quad -\alpha < y < \beta \quad (\alpha > 0, \beta > 0) \quad (2a)$$

or in one of the half-planes

$$-\infty < x < +\infty, \quad -\infty < y < \beta \quad (\beta > 0), \quad (2b)$$

$$-\infty < x < +\infty, \quad -\alpha < y < +\infty \quad (\alpha > 0). \quad (2c)$$

Problem 1. In the domain (2), find a solution of equation (1) satisfying one of the boundary conditions

$$U(x, y)|_{y=\beta} = F_1(x) \quad (-\infty < x < +\infty, \beta \neq \infty) \quad (3a)$$

or

$$U(x, y)|_{y=-\alpha} = F_2(x) \quad (-\infty < x < +\infty, \alpha \neq \infty), \quad (3b)$$

and the “gluing condition”

$$\lim_{y \rightarrow +0} y^{-n-1} U(x, y) = \lim_{y \rightarrow -0} y^{-n-1} U(x, y). \quad (4)$$

Problem 2. Assuming that $F_1(x)$ and $F_2(x)$ are periodic functions of x with period T , find in the domain (2) a solution of equation (1) satisfying one of the conditions (3), condition (4), and periodicity in x with period T . By a solution we mean a function satisfying equation (1) in the domain (2) for $y \neq 0$.

Problem 1 is solved with the aid of the two-sided Laplace transform and its inversion formula (1).

We shall say that a function $F(x, y)$ belongs to the class $E(\lambda, \mu)$ on some interval of variation of y if, for every y from this interval, it is defined for almost all values of x ($-\infty < x < +\infty$), is absolutely integrable on every finite interval of variation of x , and satisfies the estimates

$$|F(x, y)| \leq M e^{\lambda x} \quad \text{as } x \rightarrow +\infty, \quad |F(x, y)| \leq M e^{\mu x} \quad \text{as } x \rightarrow -\infty \quad (5)$$

uniformly with respect to y in any finite part of the interval of variation of y . Here $M = \text{const}$, and λ and μ are real numbers, with $\lambda < \mu$.

This definition is preserved (with the obvious changes) for a function depending only on x .

The following existence theorem for a solution holds:

Theorem. For the existence of a solution of problem 1 it is sufficient that:

- a) in the case of the boundary condition (3a), the function $F_1(x) \in E(\lambda, \mu)$ and, for $-\infty < x < +\infty$, possess almost everywhere a derivative $F_1'(x) \in E(\lambda, \mu)$, that $F_1'(x)$ be a function of bounded variation for $X_1 \leq x \leq X_2$ ($X_1 \leq X_2$), and that it satisfy the estimates

$$|F_1'(x+h) - F_1'(x)| \leq \begin{cases} M|h|^\nu e^{\lambda x} & \text{for } x \geq X_2, \\ M|h|^\nu e^{\mu x} & \text{for } x \leq X_1, \end{cases} \quad (0 < \nu \leq 1); \quad (6)$$

- b) in the case of the boundary condition (3b), the function $F_2(x)$ admit a continuation to complex values x ($x = \xi + i\eta$), be regular in the strip $|\eta| \leq \Phi(\beta)$, $-\infty < \xi < +\infty$, and satisfy in this strip the estimates

$$|F_2(x)| \leq M e^{\lambda \xi} \quad \text{as } \xi \rightarrow +\infty, \quad |F_2(x)| \leq M e^{\mu \xi} \quad \text{as } \xi \rightarrow -\infty \quad (7)$$

uniformly with respect to η . Here

$$\Phi(y) = \int_0^{2\sqrt{y}} y^{m/2} \sqrt{a(y)} dy \quad \left(\arg \sqrt{y} = \frac{1}{2} \arg y \right). \quad (8)$$

Let $f_j(x)$ denote the images of $F_j(x)$ under the bilateral Laplace transform:

$$f_j(x) = \int_{-\infty}^{+\infty} F_j(x) e^{-px} dx \quad (p = s + i\sigma). \quad (9)$$

Let $u(p, y)$ be a solution of the ordinary differential equation

$$\frac{d^2 u}{dy^2} + \left\{ -\frac{n}{y} + b(y) \right\} \frac{du}{dy} + \{ p^2 y^m a(y) + c(y) \} u = 0 \quad (-\alpha < y < \beta), \quad (10)$$

satisfying one of the boundary conditions

$$u(p, y)|_{y=\beta} = f_1(p) \quad (11a)$$

or

$$u(p, y)|_{y=-\alpha} = f_2(p) \quad (11b)$$

(respectively in the cases (3a) and (3b)) and the condition

$$\lim_{y \rightarrow +0} y^{-n-1} u(p, y) = \lim_{y \rightarrow -0} y^{-n-1} u(p, y). \quad (12)$$

Such a solution exists and is unique. It is equal to $f_1(p)u_1(p, y)/u_1(p, \beta)$ in the case (11a) and to $f_2(p)u_1(p, y)/u_1(p, -\alpha)$ in the case (11b), where $u_1(p, y)$ is the solution of equation (10) satisfying condition (12).

If in the strip $\lambda < \operatorname{Re} p < \mu$ of the complex p -plane there are singularities (poles) $p_k = s_k + i\sigma_k$ ($k = 1, 2, \dots, l-1$) of the function $u(p, y)$ such that $s_{k-1} < s_k$, then, under the above sufficient conditions, there exist l ($l \geq 1$) solutions of problem 1:

$$U_k(x, y) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} u(p, y) e^{px} dp \quad (s_{k-1} < c < s_k), \quad (13)$$

where $k = 1, 2, \dots, l$, $s_0 = \lambda + \varepsilon$, $s_l = \mu - \varepsilon$, $\varepsilon > 0$ and arbitrarily small.

For the proof, asymptotic expressions for linearly independent solutions of equation (10) as $|p| \rightarrow \infty$, obtained by Langer (2), are used.

The solutions $U_k(x, y)$ satisfy, as $x \rightarrow \pm\infty$, the conditions

$$y^{-n-1} U_k(x, y) \in E(s_{k-1}, s_k) \quad (14a)$$

($-\alpha \leq y \leq \beta$, $-\infty < y \leq \beta$, $-\alpha \leq y < +\infty$ respectively in cases (2a), (2b), and (2c)),

$$\frac{\partial U_k}{\partial x} \in E(s_{k-1}, s_k), \quad \frac{\partial U_k}{\partial y} \in E(s_{k-1}, s_k), \quad \frac{\partial^2 U_k}{\partial y^2} \in E(s_{k-1}, s_k) \quad (14b)$$

($-\alpha + \varepsilon \leq y \leq \beta - \varepsilon$, $-\infty < y \leq \beta - \varepsilon$, $-\alpha + \varepsilon \leq y < +\infty$ respectively in cases (2a), (2b), and (2c)).

The following theorem on the uniqueness of the solution of problem 1 is valid:

Theorem. *Each of the solutions $U_k(x, y)$ of problem 1 is unique in the class of functions satisfying conditions (14).*

The validity of this theorem follows from the fact that, if a solution of problem 1 satisfying conditions (14) exists, then it can be represented by the integral (13).

Finally, it is easy to verify the validity of the following theorem on the continuous dependence of the solution on the boundary value.

Theorem. *Let $\bar{F}_j(x)$, $\tilde{F}_j(x)$ ($j = 1$ or 2) be functions satisfying sufficient conditions for the existence of a solution, and let $\bar{U}_k(x, y)$, $\tilde{U}_k(x, y)$ be the corresponding solutions. Then for arbitrarily small $\varepsilon > 0$ there exists a $\delta > 0$ such that*

$$\left| \frac{\partial^p \bar{U}_k(x, y)}{\partial x^q \partial y^{p-q}} - \frac{\partial^p \tilde{U}_k(x, y)}{\partial x^q \partial y^{p-q}} \right| \leq \begin{cases} \varepsilon e^{s_{k-1}x}, & \text{for } x \geq 0, \\ \varepsilon e^{s_k x}, & \text{for } x < 0, \end{cases} \quad (15)$$

$$p = 0, 1, 2; \quad q = 0, 1, 2; \quad k = 1, 2, \dots, l,$$

provided that

$$|\bar{F}_j(x) - \tilde{F}_j(x)| \leq \delta \quad (16)$$

uniformly with respect to x in any finite interval of variation of x .

If the interval of variation of y is infinite, then in order for the last theorem to be valid it is necessary to assume additionally that the coefficients of equation (1) satisfy certain additional conditions as $y \rightarrow \infty$.

The estimates (15), generally speaking, are not uniform in x . However, if $s_{k-1} \leq 0$ and $s_k \geq 0$ ($s_{k-1} < s_k$), then the continuous dependence of the solution on the boundary value is uniform in x for $-\infty < x < +\infty$.

Remark 1. In addition to singularities coinciding with the singularities of $f_j(p)$, which lie outside the strip $\lambda < \operatorname{Re} p < \mu$, the function $u(p, y)$ may have poles coinciding with the zeros of $u_1(p, \beta)$ in case (11a) and of $u_1(p, -\alpha)$ in case (11b). The function $u_1(p, \beta)$ has an asymptotically equidistant infinite sequence of zeros on both sides of the real axis of the complex p -plane. In addition, it may have no more than a finite number of imaginary complex zeros lying on the imaginary axis. If the condition

$$-\frac{in(n+2)}{4y^2} + \frac{nb(y)}{2y} - \frac{1}{4}b^2(y) - \frac{1}{2}b'(y) + c(y) \leq 0 \quad \text{for } 0 \leq y \leq \beta, \quad (17)$$

is satisfied, then there are no imaginary complex zeros. For the function $u_1(p, -\alpha)$ the axes of the p -plane interchange roles (condition (18) must be satisfied for $-\alpha \leq y \leq 0$).

Remark 2. If the imaginary axis of the complex p -plane is free of singularities of $u(p, y)$, then the theorems on existence, uniqueness, and stability of the solution will, with the corresponding changes, remain valid also in the case where the exponential estimates are replaced everywhere by estimates of the form $M/|x|^\chi$ ($\chi > 1$). The proof is based on the application of the Fourier transform. ($s = 0$ in formula (9), $c = 0$ in formula (13), and it defines one solution, $s_{k-1} = s_k = 0$ in the estimates (15).)

Problem 2 is solved by the Fourier method. Let $v_k(y)$, $w_k(y)$ be solutions of equation (10) for $p = i2k\pi/T$ ($k = 0, 1, 2, \dots$ for $v_k(y)$, $k = 1, 2, 3, \dots$ for $w_k(y)$), satisfying condition (12) and one of the boundary conditions

$$v_k(y)|_{y=\beta} = q_k^{(1)}, \quad w_k(y)|_{y=\beta} = r_k^{(1)}, \quad (18a)$$

or

$$v_k(y)|_{y=-\alpha} = q_k^{(2)}, \quad w_k(y)|_{y=-\alpha} = r_k^{(2)}, \quad (18b)$$

where $q_k^{(j)}$, $r_k^{(j)}$ are the Fourier coefficients of the function $F_j(x)$ ($j = 1, 2$), respectively with respect to cosines and sines.

Such solutions exist and are unique, except for isolated values $\beta(\alpha)$, the set of which is at most countable. For the latter values $\beta(\alpha)$, Problem 2 is posed incorrectly. The presence of such values $\beta(\alpha)$ is connected with the zeros of the function $u_1(p, \beta)$ ($u_1(p, -\alpha)$) lying on the imaginary axis of the p -plane.

If Problem 2 is correct and

- a) in the case of the boundary condition (3a): $F_1(x)$ is a periodic function having a derivative almost everywhere, $F_1'(x)$ is a function of bounded variation or satisfies a Lipschitz condition of order $0 < \nu \leq 1$;

b) in the case of the boundary condition (3b): $F_2(x)$, for $x = \xi + i\eta$, is regular in the strip $|\eta| \leq \Phi(\beta)$, $-\infty < \xi < +\infty$, and periodic in ξ with period T , then the unique solution of Problem 2 is represented by the Fourier series

$$U(x, y) = \frac{v_0(y)}{2} + \sum_{k=1}^{\infty} \left\{ v_k(y) \cos \frac{2k\pi x}{T} + w_k(y) \sin \frac{2k\pi x}{T} \right\}. \quad (19)$$

The solution depends continuously on the boundary value.

The results of the present work and the methods developed in it can be used to study infinitely small bendings³ of surfaces of revolution and of helical surfaces whose Gaussian curvature changes sign.

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Note: Figure translations are in progress. See original paper for figures.

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