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**Abstract**

**Full Text**

**HYDROMECHANICS**

**T. AITMURZAEV**

**A METHOD FOR SOLVING THE EQUATIONS OF UNSTEADY GAS FLOW WITH ALLOWANCE FOR DISSIPATIVE PROCESSES IN GENERAL RELATIVITY**

*(Presented by Academician N. N. Bogolyubov, 16 X 1956)*

Let us consider an unsteady relativistic flow of a viscous heat-conducting gas inside gravitating masses. As is known, the equations of motion and the energy equation are contained in the equations

$$\partial T^{ik} / \partial x^k + \Gamma_{km}^i T^{mk} + \Gamma_{km}^k T^{im} = 0. \quad (1)$$

In our case the energy-momentum tensor of the material medium, which itself generates the gravitational field, is

$$T^{ik} = w u^i u^k - p g^{ik} + \tau^{ik}. \quad (2)$$

Here  $w$  is the heat function;  $p$  is the pressure;  $u^i$  is the velocity of energy motion.

In the case of a viscous heat-conducting gas in relativistic mechanics, as the macroscopic velocity we shall take the velocity of energy motion, and not the velocity of the matter flux. The frictional stress tensor  $\tau^{ik}$  is proportional to the strain-rate tensor and is determined in the form

$$\begin{aligned} \tau^{ik} = \eta \{ (g^{il} - u^i u^l) [u^k; e] + (g^{kl} - u^k u^l) [u^i; e] \} - \\ - (\zeta - 2/3 \eta) (u^i u^k - g^{ik}) [u^l; e], \end{aligned} \quad (3)$$

where  $\eta$ ,  $\zeta$  are the coefficients of viscosity in the ordinary nonrelativistic sense. Finally,

$$\Gamma_{kl}^i = \frac{1}{2} g^{im} (\partial g_{mk} / \partial x^l + \partial g_{ml} / \partial x^k - \partial g_{kl} / \partial x^m)$$

are the Christoffel symbols.

The equation of continuity of the matter flux is

$$(\rho u^i + \nu^i)_{;i} = 0, \quad (4)$$

where  $\rho$  is the amount of rest mass per unit volume moving together with the energy;  $\nu^i$  is the vector of the density of the heat fluxes of matter, which is due to thermal conductivity and is equal to

$$\nu^i = K \left( \frac{T}{J} \right)^2 [g^{il} - u^i u^l] \frac{\partial}{\partial x^l} \left( \frac{M}{T} \right);$$

$K$  is the coefficient of thermal conductivity in the ordinary sense;  $T$  is the absolute temperature;  $J$  is the heat content;  $M = (w - T\sigma)/\rho$  is the relativistic chemical potential;  $\sigma$  is the specific entropy.

As the third equation we take the equation of the gravitational field itself,

$$R_{ik} = \varkappa (T_{ik} - 1/2 T g_{ik}). \quad (5)$$

Here  $\varkappa$  is a constant;  $R_{ik} = \partial \Gamma_{ik}^l / \partial x^l - \partial \Gamma_{il}^k / \partial x^k + \Gamma_{ik}^l \Gamma_{lr}^r - \Gamma_{il}^r \Gamma_{kr}^l$  is the Ricci tensor;  $T = g^{ik} T_{ik}$  is a scalar.

In what follows, in order to solve the questions of interest to us, we shall consider a spherically symmetric gravitational field inside gravitating

masses in a coordinate system moving together with the energy. In this case one may speak of a local Lorentz reference system.

From these assumptions it follows that:

- a) there exists only a radial velocity, but with our choice of coordinates it is equal to zero:

$$u^1 = 0; \quad (6)$$

- b) all components of the fundamental tensor with unlike indices are equal to zero, i.e.

$$g^{01} = 0. \quad (7)$$

We call conditions (6) and (7) coordinate conditions.

We choose the expression for the interval  $ds$  in the form

$$ds^2 = -e^\lambda dr^2 - r^2 e^\gamma (d\vartheta^2 + \sin^2 \vartheta d\varphi^2) + e^\nu dt^2, \quad (8)$$

where  $\lambda, \gamma, \nu$  are functions of  $r, t$ . Since  $u^\alpha = 0$  (where  $\alpha = 1, 2, 3$ ), it follows from the condition  $u_i u^i = 1$  that  $u^0 = 1/\sqrt{e^\nu}$ ;  $u_0 = \sqrt{e^\nu}$  is the zero velocity.

It is now easy to write explicitly the expression for  $\Gamma_{kl}^i$ .

Components of the friction tensor\*

$$\begin{aligned}\tau^{11} &= -\eta\dot{\lambda}\frac{e^{-\lambda}}{\sqrt{e^\nu}} - \left(\zeta - \frac{2}{3}\eta\right) \left(\frac{\dot{\lambda}}{2} + \dot{\gamma}\right) \frac{e^{-\lambda}}{\sqrt{e^\nu}}; \\ \tau^{22} &= -\eta\dot{\gamma}\frac{r^{-2}e^{-\gamma}}{\sqrt{e^\nu}} - \left(\zeta - \frac{2}{3}\eta\right) \left(\frac{\dot{\lambda}}{2} + \dot{\gamma}\right) \frac{r^{-2}e^{-\gamma}}{\sqrt{e^\nu}}; \\ \tau^{33} &= -\eta\dot{\gamma}\frac{r^{-2}\sin^{-2}\vartheta e^{-\gamma}}{\sqrt{e^\nu}} - \left(\zeta - \frac{2}{3}\eta\right) \left(\frac{\dot{\lambda}}{2} + \dot{\gamma}\right) \frac{r^{-2}\sin^{-2}\vartheta e^{-\gamma}}{\sqrt{e^\nu}}.\end{aligned}\quad (9)$$

Consequently, the scalar quantity  $T = g^{ik}T_{ik}$  will be

$$T = \varepsilon - 3p, \quad (10)$$

where  $\varepsilon$  includes the internal thermal energy and the residual energy of the particles themselves.

It is known that the gravitational equation includes the equation of the law of conservation of energy-momentum. For our case, when the motions of a viscous heat-conducting gas occur inside gravitating masses (the Sun, stars, and planets) with spherical symmetry in a coordinate system moving together with the energy, we obtain from (1), (4), and (5) the following independent system of 5 equations with 5 unknown functions  $\varepsilon, \rho, \lambda, \gamma, \nu$  of  $r, t$ :

$$\begin{aligned}\dot{\varepsilon} + w(\dot{\lambda}/2 + \dot{\gamma}) &= \frac{1}{3}\eta e^{-\nu/2}(\dot{\lambda} - \dot{\gamma})^2; \\ wp'/2 &= -p' + \eta e^{-\nu/2}(\gamma' + 2/r)(\dot{\lambda} - \dot{\gamma}) + e^{-\nu/2} \left[\frac{2}{3}\eta(\dot{\lambda} - \dot{\gamma})\right]'; \\ e^{-\nu/2} \left[\dot{\rho} + \rho \left(\frac{\dot{\lambda}}{2} + \dot{\gamma}\right)\right] &= e^{-\lambda} \left[K \left(\frac{T}{J}\right)^2 \left(\frac{M}{T}\right)'\right] \left(\frac{\nu'}{2} - \frac{\lambda'}{2} + \gamma' + \frac{2}{r}\right) + \\ &+ e^{-\lambda} \left[K \left(\frac{T}{J}\right)^2 \left(\frac{M}{T}\right)'\right]'; \\ e^{-\lambda} \left[-\gamma'' - \frac{3}{4}\gamma'^2 + \frac{\lambda'\gamma'}{2} - \frac{1}{r}(3\gamma' - \lambda') - \frac{1}{r^2}\right] &+ \frac{1}{r^2}e^{-\gamma} + \\ &+ \frac{1}{2}e^{-\nu} \left[\dot{\lambda}\dot{\gamma} + \frac{\dot{\gamma}^2}{2}\right] = \chi\varepsilon; \\ e^{-\lambda} \left[-\frac{\nu''}{2} - \gamma'' - \frac{\gamma'^2}{4} + \frac{\nu'\lambda'}{4} + \frac{\lambda'\gamma'}{2} - \frac{1}{r}(2\gamma' - \lambda')\right] &+ \end{aligned}\quad (A)$$

$$+e^{-\nu} \left[ \frac{\ddot{\lambda}}{2} + \frac{\dot{\lambda}^2}{4} - \frac{\dot{\nu}\dot{\lambda}}{4} + \frac{\dot{\lambda}\dot{\gamma}}{2} \right] = \frac{\chi}{2}(\varepsilon - p) - \frac{2\chi}{3}\eta e^{-\nu/2}(\dot{\lambda} - \dot{\gamma}).$$

\* Primes denote differentiation with respect to  $r$ , and dots over a letter denote differentiation with respect to  $t$ .

Here we have set  $\zeta = 0$  everywhere, treating the gas as a mixture of an ideal gas and a photon gas.

We shall seek the solution of system (A) in the form of a power series in even functions of the spherical coordinate, since all functions entering system (A), both thermodynamic functions and logarithms of the gravitational potentials, are distributed symmetrically about the origin. We prescribe the initial conditions on the  $r$ -axis:

$$\varepsilon(r, 0) = \sum \varepsilon_{i0} r^{2i}, \quad \rho(r, 0) = \sum \rho_{i0} r^{2i} \quad (11)$$

and set

$$\lambda(r, 0) = 0; \quad (12)$$

$$\dot{\lambda}(r, 0) = \sum \lambda_{i1} r^{2i}; \quad (13)$$

on the  $t$ -axis we set

$$\gamma(0, t) = 0. \quad (14)$$

Conditions (12) and (14) show that the radial coordinate  $r$  and the time coordinate are fixed.

We solve system (A) in a neighborhood of the center of the gravitating masses for  $t$  close to zero. Then the difference  $\lambda - \gamma$  on the  $t$ -axis gives a constant value for all values of  $t$ . Geometrically this means that at the center of the gravitating masses there is no difference between the radial and transverse stretching coefficients, i.e.

$$\lambda(0, t) = \gamma(0, t). \quad (15)$$

Using conditions (11)–(15) and l' Hôpital' s rule in passing to the limit as  $r \rightarrow 0$ ,  $t \rightarrow 0$  at the center of the gravitating masses, from system (A) we obtain the first, second, and third approximations.

Let us write the third approximation for the functions  $(\lambda, \gamma)$ , and the second for  $\nu$ , where the thermodynamic functions are likewise computed with accuracy up to terms of second order of smallness:

$$\dot{\varepsilon} = -^3/2 w \dot{\lambda}; \quad \ddot{\varepsilon} = -^3/2 w \dot{\lambda} - ^3/2 w \ddot{\lambda}; \quad \varepsilon'' = -w(\lambda''/2 + \gamma'') - ^3/2 w'' \dot{\lambda};$$

$$\dot{\rho} = 3K(T/J)^2(M/T)'' - ^3/2 \rho \dot{\lambda};$$

$$\rho'' = 5K''(T/J)^2(M/T)'' + 10K(T/J)(T/J)''(M/T)'' + ^5/3 K(T/J)^2(M/T)^{IV} +$$

$$+ K(T/J)^2(M/T)''(\gamma'' + 2\gamma'') + ^1/2 \gamma''(\dot{\rho} + ^3/2 \rho \dot{\lambda}) - \rho(\lambda''/2 + \gamma'') - ^3/2 \rho'' \dot{\lambda};$$

$$\ddot{\rho} = -3\dot{\lambda}K(T/J)^2(M/T)'' + 3\dot{K}(T/J)^2(M/T)'' +$$

$$+ 6K(T/J)(T/J)'(M/T)'' + 3K(T/J)^2(M/T)'' - ^3/2(\rho \ddot{\lambda} + \dot{\rho} \dot{\lambda});$$

$$\gamma'' = -\frac{18}{9w + 10\eta \dot{\lambda}} p'' + \frac{20}{9w + 10\eta \dot{\lambda}} \eta \dot{\lambda}'';$$

$$\gamma^{IV} = -\frac{30}{15w + 14\eta \dot{\lambda}} p^{IV} - \frac{45}{15w + 14\eta \dot{\lambda}} w'' \gamma'' + \frac{210}{15w + 14\eta \dot{\lambda}} \gamma''(\lambda'' - \gamma'') -$$

$$-\frac{75}{15w + 14\eta \dot{\lambda}} \eta \gamma''(\dot{\lambda}'' - \dot{\gamma}'') + \frac{6}{15w + 14\eta \dot{\lambda}} \eta \gamma''(\dot{\lambda}'' - 8\dot{\gamma}'') +$$

$$+ \frac{14}{5(15w + 14\eta \dot{\lambda})} \eta \dot{\lambda}^2(18\lambda'' - 19\gamma'') - \frac{42}{5(15w + 14\eta \dot{\lambda})} \eta \dot{\lambda}(\dot{\lambda}'' + 2\dot{\gamma}'') +$$

$$+ \frac{28}{15w + 14\eta \dot{\lambda}} \eta \dot{\lambda}^{IV} - \frac{413}{3(15w + 14\eta \dot{\lambda})} \eta \dot{\lambda} \dot{\gamma}^{IV} - \frac{14}{15w + 14\eta \dot{\lambda}} \dot{\gamma} \lambda''^2 -$$

$$-\frac{49}{15w + 14\eta \dot{\lambda}} \eta \dot{\lambda} \gamma''^2 - \frac{28}{5(15w + 14\eta \dot{\lambda})} \eta \dot{\lambda} \dot{\lambda}^2 + \frac{56}{5(15w + 14\eta \dot{\lambda})} \eta \dot{\lambda} \lambda \gamma'' -$$

$$-\frac{84}{5(15w + 14\eta \dot{\lambda})} \eta \dot{\lambda} \dot{\lambda} \gamma'' - \frac{77}{5(15w + 14\eta \dot{\lambda})} \eta \dot{\lambda} 3\gamma'' + \frac{252}{5(15w + 14\eta \dot{\lambda})} \eta \dot{\lambda} 3\gamma'' +$$

$$\begin{aligned}
 & + \frac{84}{5(15w + 14\eta\dot{\lambda})} \eta\dot{\lambda}3\dot{\lambda} - \frac{28\kappa}{5(15w + 14\eta\dot{\lambda})} \eta\dot{\lambda}^2(\dot{\varepsilon} - \dot{p}) + \frac{56\kappa}{15(15w + 14\eta\dot{\lambda})} \eta\dot{\lambda}\dot{\varepsilon} + \\
 & + \frac{84\kappa}{5(15w + 14\eta\dot{\lambda})} \eta\varepsilon'' - \frac{14\kappa}{15w + 14\eta\dot{\lambda}} \eta\dot{\lambda}(\varepsilon'' - p'') + \frac{56\kappa}{3(15w + 14\eta\dot{\lambda})} \eta^2\dot{\lambda}(\lambda'' - \gamma'');
 \end{aligned}$$

$$\begin{aligned}
 \ddot{\gamma}'' & = -\frac{2}{w}\dot{p}'' - \frac{\dot{w}}{w}\dot{\gamma}'' + \frac{10}{3w}\dot{\eta}(\lambda'' - \gamma'') - \frac{140}{9w}\eta\dot{\lambda}\dot{\gamma}'' + \frac{20}{9w}\eta\gamma^{IV} \\
 & + \frac{440}{27w}\eta\gamma^{IV} + \frac{20}{9w}\eta\gamma''^2 + \frac{40}{9w}\eta\gamma''^2 + \frac{20}{9w}\eta\dot{\lambda}\dot{\gamma}'' - \frac{10}{3w}\eta\dot{\lambda}\dot{\gamma}'' \\
 & + \frac{40}{9w}\eta\lambda^2\dot{\gamma}'' + \frac{10}{w}\eta\lambda^2\dot{\gamma}'' + \frac{10}{3w}\eta\lambda\ddot{\lambda} - \frac{10}{9w}\eta\dot{\lambda}^2 - \frac{10\kappa}{9w}\eta\dot{\lambda}(\dot{\varepsilon} - \dot{p}) \\
 & + \frac{20\kappa}{27w}\dot{\eta}\dot{\varepsilon} + \frac{20\kappa}{9w}\eta(\varepsilon'' - p'') - \frac{80\kappa}{27w}\eta^2(\lambda'' - \gamma'');
 \end{aligned}$$

$$\ddot{\lambda} = \gamma'' - \dot{\lambda}^2/2 - \frac{1}{3}\kappa(\varepsilon + 3p); \quad \ddot{\lambda} = \dot{\gamma}'' + 6\dot{\gamma}'' - \dot{\lambda}\gamma'' - 6\dot{\lambda}\gamma'' - 2\ddot{\lambda}'' - 3\dot{\lambda}\ddot{\lambda} + \kappa(\dot{\varepsilon} - \dot{p});$$

$$\ddot{\lambda}'' = \gamma^{IV} + \frac{22}{3}\gamma^{IV} + \gamma''^2 + 2\gamma''^2 + \gamma'' \left( \ddot{\lambda} + \frac{3}{2}\dot{\lambda}^2 \right) + \dot{\lambda}(\dot{\gamma}''/2 - 2\dot{\lambda}'' - \dot{\gamma}'') + \kappa(\varepsilon'' - p'') - \frac{4}{3}\kappa\eta(\lambda'' - \gamma'');$$

$$\gamma'' = \lambda^2/6 - \frac{2}{9}\kappa\varepsilon; \quad \dot{\gamma}'' = \frac{1}{3}(\dot{\lambda}'' + \dot{\lambda}\gamma'');$$

$$\ddot{\gamma}'' = \gamma''(\ddot{\lambda} - \dot{\lambda}^2) + \frac{1}{3}(\dot{\lambda}\dot{\lambda} + \ddot{\lambda}'' + \dot{\lambda}^2) - 2\dot{\lambda}(\dot{\lambda}''/3 - \dot{\gamma}'') - \frac{2}{9}\kappa\dot{\varepsilon};$$

$$\dot{\gamma}^{IV} = \frac{6}{5}\gamma''(2\dot{\lambda}'' - \dot{\gamma}'') + \frac{6}{25}\dot{\lambda}(\lambda'' + 2\gamma'') + \frac{6}{25}\lambda(\dot{\lambda}'' + 2\dot{\gamma}'') + \dot{\lambda}^{IV}/5 + \dot{\lambda}\gamma^{IV} + \frac{3}{5}\dot{\lambda}\gamma''^2 - \frac{9}{25}\dot{\lambda}^2\gamma'' - \frac{18}{25}\dot{\lambda}\ddot{\lambda}\gamma'' - \frac{12}{25}\kappa\dot{\varepsilon}'';$$

$$\gamma^{IV} = \frac{6}{25}(\lambda\lambda'' + 2\gamma''\lambda - 2\kappa\varepsilon'') - \frac{3}{5}\gamma''^2 - \frac{9}{25}\lambda^2\gamma'';$$

$$\gamma^{VI} = \frac{5}{13}\lambda(\lambda^{IV} + 2\gamma^{IV}) - \frac{30}{13}\lambda''(\lambda'' + 2\gamma'') - \frac{15}{26}\lambda^2\gamma^{IV} + \frac{45}{26}\lambda^2\gamma''^2 - \frac{105}{13}\gamma''\gamma^{IV} + \frac{45}{13}\gamma''^3 + \frac{30}{13}\lambda\gamma'' + \frac{15}{13}\gamma''^2 - \frac{10}{13}\kappa\varepsilon^{IV}.$$

The results obtained can be applied in astrophysics and in plasma physics.

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*Note: Figure translations are in progress. See original paper for figures.*

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