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Abstract

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THE CAUCHY PROBLEM FOR SYSTEMS OF LINEAR PARTIAL DIFFERENTIAL EQUATIONS WITH WEIGHT GREATER THAN ONE

(Presented by Academician I. G. Petrovskii, 29 IV 1957)

1. In the present paper we consider the Cauchy problem for the system

$$\frac{\partial^{n_i} u_i}{\partial t^{n_i}} = \sum_{(k_s)} \sum_{j=1}^{p} A_{ij}^{(k_0,k_1,\dots,k_n)}(t) \frac{\partial^{k_0+k_1+\dots+k_n} u_j}{\partial t^{k_0} \partial x_1^{k_1} \cdots \partial x_n^{k_n}} \qquad (i = 1, 2, \dots, p) \qquad (1)$$

with initial conditions

$$\left.\frac{\partial^k u_i}{\partial t^k}\right|_{t=0} = \varphi_i^{(k)}(x_1, x_2, \dots, x_n) \qquad (i = 1, 2, \dots, p;\ k = 1, 2, \dots, n_i - 1), \qquad (2)$$

where $\sum_{(k_s)}$ denotes summation over all $k_0, k_1, \dots, k_n$ for which

$$\sum_{s=0}^{n} k_s \leq L;$$

L is some fixed positive integer;

$$k_0 < n_j;$$

$A_{ij}^{(k_0,k_1,\dots,k_n)}(t)$ and $\varphi_i^{(k)}(x_1, x_2, \dots, x_n)$ are complex functions of real arguments.

In what follows we shall consider the equations

$$\left\{ \left\| \sum_{(k_s,\beta)} A^{(k_0,k_1,\dots,k_n)}(t) \lambda^{k_0} (i\alpha_1)^{k_1} \cdots (i\alpha_n)^{k_n} \right\| - \left\| \begin{matrix} \lambda^{n_1} & & 0 \\ & \lambda^{n_2} & \\ 0 & & \ddots \\ & & & \lambda^{n_p} \end{matrix} \right\| \right\} = 0, \qquad (3)$$

where $\alpha_1, \alpha_2, \dots, \alpha_n$ are real parameters; $\sum_{(k_s,\beta)}$ denotes summation over all $k_0, k_1, \dots, k_n$ for which

$$k_0\sigma + \sum_{s=1}^{n} k_s \geq n_j\sigma - \beta;$$

the number σ is the weight of the system (1), i.e. the least of the numbers satisfying the inequalities

$$k_0\sigma + \sum_{s=1}^{n} k_s \leq n_j\sigma \qquad (j = 1, 2, \dots, p),$$

$\beta = 0, 1, 2, \dots, \sigma - 1$.

Equations (3) will be called **characteristic**.

Everywhere in what follows only such systems are considered for which the roots of the characteristic equation (3), for $\beta = 0$,

$$\sum_{k=1}^{n} \alpha_k^2 = 1$$

and $0 \leq t \leq T$, for some $T > 0$, are not all equal to zero and their real parts are nonpositive. Hence it follows that σ is an integer (see (1), p. 119).

We reduce the system (1) to a system of partial differential equations of first order with respect to t by replacing

$$\partial^k u_i / \partial t^k \qquad (i = 1, 2, \dots, p;\ k = 1, 2, \dots, n_i - 1)$$

by new functions $u_{p+1}, u_{p+2}, \dots, u_N$:

$$\frac{\partial u_i}{\partial t} = \sum_{(k_s)} \sum_{j=1}^{N} A^{(k_0,k_1,\dots,k_n)}(t) \frac{\partial^{k_1+k_2+\dots+k_n} u_j}{\partial x_1^{k_1} \partial x_2^{k_2} \cdots \partial x_n^{k_n}} \quad (i = 1, 2, \dots, N). \tag{4}$$

The corresponding initial conditions are written in the form

$$u_i\big|_{t=0} = \varphi_i(x_1, x_2, \dots, x_n) \quad (i = 1, 2, \dots, N). \tag{5}$$

After the Fourier transform, system (4) takes the form

$$\frac{dv_i}{dt} = \sum_{(k_s)} \sum_{j=1}^{N} A_{ij}^{(k_0,k_1,\dots,k_n)}(t)(i\alpha_1)^{k_1}(i\alpha_2)^{k_2} \cdots (i\alpha_n)^{k_n} v_j \quad (i = 1, 2, \dots, N). \tag{6}$$

By studying the fundamental solution of system (6)

$$v_i^{(l)}\Big|_{t=0} = \begin{cases} 0, & i \neq l, \\ 1, & i = l, \end{cases} \tag{7}$$

the following theorems are proved.

Theorem 1. Let there exist $\alpha_0 > 0$, $0 \leq \beta \leq \sigma - 1$, $c > 0$ such that, for this β, $0 \leq t \leq T$ and $\sum_{k=1}^{n} \alpha_k^2 \geq \alpha_0^2$, the real parts of all roots of the characteristic equation (3) are less than $-c\alpha^{\sigma-\beta}$, where $\alpha^2 = \sum_{k=1}^{n} \alpha_k^2$. Then there exist constants $c_1 > 0$, $c_2 > 0$ such that

$$\left|v_i^{(l)}(t, \alpha_1, \alpha_2, \dots, \alpha_n)\right| \leq c_1 \alpha^{\sigma(M-1)} e^{-c_2\alpha^{\sigma-\beta}},$$

where $M = \max n_i$.

In this case, analogously to what was done by I. G. Petrovskii ([2], Ch. 2, § 4) for parabolic systems, it is proved that the solution of the Cauchy problem (1)–(2) behaves like the solution of the Cauchy problem for parabolic systems, i.e., it is analytic in $(x_1, x_2, \dots, x_n)$ for $t > 0$, if the initial functions $\varphi_i^{(k)}(x_1, x_2, \dots, x_n)$ are sufficiently smooth and do not grow rapidly at infinity.

G. E. Shilov ([3], p. 94) considered the special case of such systems when $n_i = 1$ $(i = 1, 2, \dots, p)$ and $A^{(k_0, k_1, \dots, k_n)}(t)$ are constant.

Theorem 2. Let the matrix of the characteristic equation (3) for $\beta = 0$ have the form

$$\begin{Vmatrix} M_1 & 0 & & \\ 0 & M_2 & & \\ & & \ddots & \\ & & & M_k \end{Vmatrix},$$

where the roots of each equation $|M_j| = 0$ are purely imaginary and distinct for $0 \leq t \leq T$ and $\sum_{k=1}^{n} \alpha_k^2 = 1$. Suppose there exists an $\alpha_0 > 0$ such that, for $\beta = \sigma - 1$, $0 \leq t \leq T$, and $\sum_{k=1}^{n} \alpha_k^2 \geq \alpha_0^2$, all roots of the characteristic equation (3) are purely imaginary.

Then problem (1), (2) is uniformly well posed and the known condition A of Petrovskii ([2], p. 3) is satisfied.

This theorem generalizes a result of S. A. Galpern ([1], p. 119), who considered the case where $A_{ij}^{(k_0, k_1, \dots, k_n)}(t) \equiv 0$ when

$$(n_j - 1)\sigma < k_0\sigma + \sum_{s=1}^{n} k_s < n_j\sigma.$$

Theorem 3. Suppose there exist $\alpha_0 \geqslant 0$, $0 \leqslant \beta \leqslant \sigma - 1$, $c > 0$ such that for this β, $0 \leqslant t \leqslant T$ and $\sum_{k=1}^{n} \alpha_k^2 \geqslant \alpha_0^2$, the real part of at least one root of the characteristic equation (3) is greater than $c\alpha^{\sigma-\beta}$.

Then the Cauchy problem (1)–(2) is posed incorrectly.

The condition of this theorem is analogous to Petrovskii' s condition B (([2]), p. 51).

2. **Theorem 4.** Suppose the following conditions are satisfied:

a) for $\beta = 0$, $0 \leqslant t \leqslant T$ and $\sum_{k=1}^{n} \alpha_k^2 = 1$, the roots of the characteristic equation (3) are purely imaginary, distinct, and do not vanish;

b) there exists an $\alpha_0 \geqslant 0$ such that for $\beta = \sigma - 1$, $0 \leqslant t \leqslant T$ and $\sum_{k=1}^{n} \alpha_k^2 \geqslant \alpha_0^2$, all roots of the characteristic equation (3) are purely imaginary;

c) the weight of the system $\sigma > 1$;

d) the functions $A_{ij}^{(k_0,k_1,\dots,k_n)}(t)$ are twice continuously differentiable;

e) the functions $\varphi_i^{(k)}(x_1, x_2, \dots, x_n)$ are continuous and absolutely integrable over the whole space $(x_1, x_2, \dots, x_n)$, as are all their partial derivatives with respect to the various combinations $(x_1, x_2, \dots, x_n)$ up to order

$$2\nu = 2 + 2\,E\left[\frac{n + \sigma(M-1)}{2}\right],$$

where $M = \max n_i$, $E[m]$ is the integer part of m.

There exist positive constants c_1, c_2, and ε such that

$$|\Delta^\nu \varphi(x_1, x_2, \dots, x_n)| \leqslant c_1 e^{-c_2 |x|^{\frac{\sigma}{\sigma-1}+\varepsilon}}, \tag{8}$$

where $\Delta^\nu \varphi$ is the polyharmonic operator of order ν applied to the function φ,

$$|x| = \left(\sum_{k=1}^{n} x_k^2\right)^{1/2}.$$

Then the solution of the Cauchy problem (1), (2) for $0 < t \leqslant T$ is an analytic function of $(x_1, x_2, \dots, x_n)$, and with respect to t has as many continuous derivatives as the functions

$$d^2 A_{ij}^{(k_0,k_1,\dots,k_n)}(t)/dt^2$$

have continuous derivatives.

The proof is carried out by studying the Green matrix (([2]), p. 34). In doing this one has to consider integrals of the form

$$\int_{\alpha_0}^{\infty} \frac{e^{i\lambda\alpha} e^{ir\alpha^{1/\sigma}}}{\alpha^m}\, d\alpha$$

as functions of r, where λ is a real parameter, $m > 1$, $\sigma > 1$, and to prove their analyticity in r. Such integrals, by means of integration along the contour in the complex plane $\alpha = \xi + i\tau$, are reduced to integrals that may be differentiated with respect to r under the integral sign.

Remark. Applying the formula given by Bochner (([4]), p. 186), we prove the following.

If the fundamental solution (7) of the system (6), as a function of the parameters $\alpha_1, \alpha_2, \dots, \alpha_n$, depends only on

$$\alpha^2 = \sum_{k=1}^{n} \alpha_k^2,$$

then in condition e) one may put

$$2\nu = 2 + 2E\left[\frac{\frac{n+2}{2} + \sigma(M-1)}{2}\right].$$

3. In the case of the Cauchy problem for the equation

$$\frac{\partial^2 u}{\partial t^2} = -\Delta^2 u \tag{9}$$

with initial conditions

$$u\big|_{t=0} = 0, \qquad \left.\frac{\partial u}{\partial t}\right|_{t=0} = \varphi(x_1, x_2, \dots, x_n) \tag{10}$$

formulas giving the solution in explicit form were found in [5]. Using these formulas, the following theorem is proved.

Theorem 5. Let the function $\varphi(x_1, x_2, \dots, x_n)$ be continuous and absolutely integrable over the whole space $(x_1, x_2, \dots, x_n)$, as are all its partial derivatives with respect to the various combinations $(x_1, x_2, \dots, x_n)$ up to order

$$2\nu = 2 + 2E\left[\frac{n-1}{4}\right].$$

Then, if $|x|^k \varphi(x_1, x_2, \dots, x_n)$, for $k = 1, 2, \dots, L_1$, where L_1 is some number, is absolutely integrable, the solution of problem (9), (10) for $t > 0$ is continuously differentiable up to order L_1.

If there exist positive constants c_1 and c_2 such that

$$|\varphi(x_1, x_2, \dots, x_n)| \le c_1 e^{-c_2|x|},$$

then the solution of problem (9), (10) for $t > 0$ is analytic in $(x_1, x_2, \dots, x_n)$.

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Note: Figure translations are in progress. See original paper for figures.