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Abstract

Full Text

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ON THE METRIC DIMENSION OF POINT SETS *

(Presented by Academician P. S. Aleksandrov on 14 IX 1956)

Metric dimension of a point set A ($\mathrm{dm}\,A$) is the least number r such that there exists an arbitrarily small shift of the set A into a locally finite polyhedron of dimension r.

In 1926 P. S. Aleksandrov proved one of the fundamental theorems of dimension theory:

Theorem on ε-shifts (1). *For every compactum Φ the equality $\dim\Phi = \mathrm{dm}\,\Phi$ holds**.*

Recently K. A. Sitnikov (2) gave an example of a set $A \subset E^3$ for which $\mathrm{dm}\,A = 1$, while $\dim A = 2$, thereby solving the problem posed by P. S. Aleksandrov in 1935. In the general case one can only assert that $\mathrm{dm}\,A \leq \dim A$. The equality $\mathrm{dm}\,A = \dim A$ also holds for those sets $A \subset E^n$ for which $\dim[A] = \dim A$ (proved by K. A. Sitnikov (2)). Hence, and from the fact that every point set can be topologically embedded in a compactum of the same dimension (Hurewicz' s theorem (3)), it follows that metric dimension is not a topological invariant. But it is preserved under so-called uniform homeomorphisms, i.e., under one-to-one mappings that are uniformly continuous in both directions. This follows easily from the following definition of metric dimension.

The **metric dimension of a metric space** R is the least number r such that for every $\varepsilon > 0$ there exists an open covering of the space R of multiplicity $\leq r+1$, each element of which has diameter less than ε.

The contents of the present work consist of the following propositions.

1. *If A is a set in the space E^n, then for the determination of the metric dimension of this set it is sufficient to consider open coverings that are locally finite in the space E^n**.*
2. *Let R be an arbitrary metric space;* $\mathrm{dm}\,R \leq r$ *if and only if into every Lebesgue covering**** of the space R one can inscribe an open covering of multiplicity less than or equal to $r+1$.*

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* Here and throughout, by point sets are meant sets of Euclidean spaces.

** Here dm denotes metric dimension, and dim the usual topological dimension, defined by means of coverings.

*** A system of sets A_γ of a space R is called **locally finite** in this space if every point of the space R has a neighborhood intersecting only a finite number of the sets A_γ.

**** An open covering λ of a metric space R is called **Lebesgue** if for some $\varepsilon > 0$ there exists a covering of the space R by sets M_γ such that the system of ε-neighborhoods $O(M_\gamma, \varepsilon)$ is inscribed in the covering λ.

3. *The metric dimension of a point set is equal to the greatest of the dimensions of the simplexes into which this set can be transformed by a uniformly continuous and essential * mapping* **.

4. *Let any P_i, Q_i $(i = 1, \dots, r+1)$ be any pairs of closed subsets of a point set A satisfying the condition $\rho(P_i, Q_i) > 0$; $\operatorname{dm} A \leq r$ if and only if there exist closed sets $\Phi_i \subset A$ satisfying the conditions: a) Φ_i separates P_i and Q_i, b)*

$$\bigcap_{i=1}^{r+1} \Phi_i = 0.$$

The theorem on essential mappings makes it possible to construct a homological characteristic of metric dimension.

Consider the set of all possible star-finite coverings of a metric space R. From it select all possible subsets in such a way that, first, each of the selected subsets contains coverings of the same multiplicity, and, second, each subset contains arbitrarily fine coverings. Assigning to each such subset the number equal to the multiplicity of the coverings contained in it, choose the subset with the smallest number. The coverings of the space R belonging to this subset will be called *basic*.

Let $M \subseteq R$. Consider a uniform ∇-cycle z of the set M, i.e. a ∇-cycle of the nerve of some Lebesgue covering λ of this set. We shall say that the cycle z can be extended to the space R, if among the basic coverings of the space R there is a covering $\alpha = \{O_\gamma\}$ satisfying the following two requirements: a) the sets $O_\gamma \cap M$ form a covering of the set M inscribed in the covering λ; b) the projection Π_α^λ of the cycle z into the complex α is extendable to a ∇-cycle of the whole complex α.

5. *If the metric dimension of the space R is equal to r, and M is an arbitrary set contained in R, then every uniform ∇-cycle of the set M of dimension greater than or equal to r can be extended to the whole space R. Conversely: if M is an arbitrary set of the space R and every uniform ∇-cycle of this set of dimension greater than or equal to r can be extended to the whole space R, then $\operatorname{dm} R \leq r$.*

The proof of this theorem rests on the following proposition.

6. *Let M be a closed subset of R; in order that a uniformly continuous mapping f of the set A into the sphere s^n be extendable to a continuous mapping of the whole space R, it is necessary and sufficient that the degree of the mapping f *** be extendable in the indicated sense to the whole space R.*

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CITED LITERATURE

[1] P. Alexandroff, *Ann. of Math.*, **30**, 101 (1929). [2] K. A. Sitnikov, *Matem. sborn.*, **37**, 3, 386 (1955). [3] W. Hurewicz, *Proc. Acad. Amsterdam*, **30**, 425 (1927).

* Essentiality of a mapping is here the ordinary one, i.e. defined by means of admissible continuous deformations.

** From this follows a characteristic of metric dimension by means of the discontinuity of extending uniformly continuous mappings into a sphere.

*** By the degree of a mapping f of a set M into a triangulation K of an n-dimensional sphere we mean the ∇-cycle constructed as follows. Let O_{e_i} be the stars of the vertices e_i of the triangulation K. Consider the covering α of the set M composed of the sets $f^{-1}(O_{e_i})$. The mapping f gives rise to a simplicial mapping of the complex α into K, which in turn determines the projection Π^k_α. The ∇-cycle $z_f = \Pi^k_\alpha \omega$, where ω is the fundamental cycle of the triangulation K, is called the degree of the mapping f of the set M into the triangulation K.

Note: Figure translations are in progress. See original paper for figures.