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Abstract

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MATHEMATICS

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ON THE DECOMPOSITION OF INFINITELY DIVISIBLE LAWS

The present note extends the results of the preceding note (1) (we shall use the terminology of note (1)).

According to the well-known formula of P. Lévy (2), an infinitely divisible (i.d.) law F has a characteristic function (c.f.) $\varphi(t)$ such that

$$\ln\varphi(t) = \beta it - \gamma t^2 + \int_{-\infty}^{0}\left(e^{itu} - 1 - \frac{itu}{1+u^2}\right)dG_-(u) + \int_{0}^{\infty}\left(e^{itu} - 1 - \frac{itu}{1+u^2}\right)dG_+(u), \tag{1}$$

where β and $\gamma \geq 0$ are real constants; $G_-(u)$ and $G_+(u)$ are nondecreasing functions such that $G_-(-\infty) = G_+(+\infty) = 0$ and

$$\int_{-a}^{0} u^2\,dG_-(u) + \int_{0}^{a} u^2\,dG_+(u) < \infty \quad \text{for every finite } a > 0;$$

$$\int_{-\varepsilon}^{0} u^2\,dG_-(u) + \int_{0}^{\varepsilon} u^2\,dG_+(u) \to 0 \quad \text{as } \varepsilon \to 0.$$

The functions $G_-(u)$ and $G_+(u)$ form the "Poisson spectrum" of the law F; $G_-(u)$ corresponds to its negative part, $G_+(u)$ to its positive part. If these functions do not have continuous components in the known decomposition of monotone functions, then the Poisson spectrum will be finite or countable. In the case of a countable spectrum we have

$$\ln\varphi(t) = \beta it - \gamma t^2 + \sum_{m=1}^{\infty}\lambda_m\left(e^{it\mu_m} - 1 - \frac{it\mu_m}{1+\mu_m^2}\right) + $$

$$+\sum_{n=1}^{\infty}\lambda_{-n}\left(e^{-it\nu_n} - 1 + \frac{it\nu_n}{1+\nu_n^2}\right), \tag{2}$$

$$\lambda_m \geq 0, \quad \lambda_n \geq 0$$

and the series

$$\sum_{m=1}^{\infty} \frac{\lambda_m \mu_m^2}{1+\mu_m^2}, \quad \sum_{n=1}^{\infty} \frac{\lambda_{-n} \nu_n^2}{1+\nu_n^2}$$

converge, and moreover

$$\sum_{\mu_m<\varepsilon} \lambda_m \mu_m^2 + \sum_{\nu_n<\varepsilon} \lambda_{-n} \nu_n^2 \to 0 \quad \text{as } \varepsilon \to 0.$$

An interesting problem in the theory of i.d. laws is the description of the class I_0 of i.d. laws having only i.d. components. This problem has not yet been solved completely, but it is possible to advance in the description of i.d. laws $F \in I_0$ having a Gaussian component ($\gamma > 0$ in (1)).

Theorem 1. In order that an infinitely divisible law F, having a Gaussian component, decompose only into infinitely divisible components, it is necessary that its Poisson spectrum be finite or countable. Moreover, the Poisson frequencies μ_m and ν_n in formula (2) (or its analogue for a finite spectrum) must coincide with the series of numbers:

for μ_m:

$$\dots,\ k_{-2}k_{-1}\mu,\ k_{-1}\mu,\ \mu,\ \frac{\mu}{k_1},\ \frac{\mu}{k_1k_2},\ \dots,\ \frac{\mu}{k_1k_2\dots k_s},\ \dots; \tag{3}$$

for ν_n:

$$\dots,\ l_{-2}l_{-1}\nu,\ l_{-1}\nu,\ \nu,\ \frac{\nu}{l_1},\ \frac{\nu}{l_1l_2},\ \dots,\ \frac{\nu}{l_1l_2\dots l_s},\ \dots, \tag{4}$$

where $\dots, k_{-2}, k_{-1}, k_1, k_2, \dots; \dots, l_{-2}, l_{-1}, l_1, l_2, \dots$ are arbitrary sets of natural numbers greater than 1 (repetitions are allowed).

In the case of bounded spectra (3) and (4), these necessary conditions are also sufficient.

In the case $\gamma = 0$ the situation changes sharply. Conditions (3) and (4) will be sufficient for a bounded spectrum, but not necessary for F to belong to I_0. In the presence of a stable, but non-Gaussian component, apparently, the law cannot belong to I_0. The classification of laws from I_0 without a Gaussian component is possible, but rather complicated and has only been begun.

The sufficiency of conditions (3) and (4) in the case of an unbounded spectrum can so far be proved only under the condition of rapid decrease of the "energy of the high frequencies," i.e. of the numbers λ_m and λ_{-n} in formula (2). In the general case the question remains open.

The theorem stated above has the property of "stability" in the sense that a law F, close (uniformly) to an infinitely divisible law with bounded Poisson spectrum of the form (3) and (4), has components close to infinitely divisible laws with the same frequencies of the Poisson spectrum.

The results obtained may be applied to the theory of summation of dependent random variables in the absence of asymptotic negligibility of the summands.

Theorem 1 admits a certain generalization in an analytic direction, useful for the theory of independent statistics (^{3}).

Theorem 2. Suppose that for some infinite sequence of real numbers tending to 0: $\{t_k\} : t_k \to 0$, the equalities

$$(\varphi_1(t_k))^{\alpha_1} \ldots (\varphi_s(t_k))^{\alpha_s} = \varphi(t_k), \tag{5}$$

hold, where $\alpha_j > 0$; $\varphi_j(t)$ are characteristic functions of random variables; $\varphi(t)$ is the characteristic function of an infinitely divisible law of the form (2) with bounded Poisson spectrum. Then each $\varphi_j(t)$ $(j = 1, 2, \ldots, s)$ will be of type (2); its Poisson spectrum will be contained in spectrum (2).

The proof of the sufficiency of conditions (3) and (4) in Theorem 1 for a bounded spectrum proceeds with the aid of the Paley-Wiener theorem on entire functions of exponential type and the author' s methods (^{3}) for the case of a bounded Poisson spectrum. An unbounded spectrum is treated for now only partially with the aid of one method of E. Fragmen. The necessity of conditions (3) and (4) follows almost directly from the following three lemmas, which are also of independent interest.

Lemma 1. Let $\alpha = p/q$ be a rational irreducible fraction and $1 < p < q$.

Construct a random variable X with characteristic function

$$\varphi(t) = E\exp(itX) = \exp(-\gamma t^2 + \lambda_1(e^{it} - 1) + \lambda_2(e^{i\alpha t} - 1)),$$

where $\gamma > 0, \quad \lambda_1 > 0, \quad \lambda_2 > 0$.

For sufficiently small $\nu > 0$ the function

$$\psi(t) = \varphi(t)\exp(-\nu(e^{it/q} - 1))$$

will be the ch. f. of some random variable Y.

We note that $\psi(t)$, evidently, does not correspond to an i. d. law. The lemma is also true for $p = 1$; then it is trivial and $\psi(t)$ corresponds to an i. d. law.

Lemma 2. Let $\alpha \in (0, 1)$ be an irrational number. Construct a random variable X with ch. f.

$$\varphi(t) = E\exp(itX) = \exp(-\gamma t^2 + \lambda_1(e^{it} - 1) + \lambda_2(e^{i\alpha t} - 1)),$$

where $\gamma > 0,\ \lambda_1 > 0,\ \lambda_2 > 0$.

For sufficiently small $\nu > 0$ and a suitably chosen small $\eta_0 > 0$, the function

$$\psi(t) = \varphi(t)\exp(-\nu(e^{\eta_0 it} - 1))$$

will be the ch. f. of some random variable Y.

It should be noted that, for almost all irrational $\alpha \in (0,1)$, any sufficiently small number $\eta_0 > 0$ may serve as the number η_0. The question of whether this is so for every irrational α is not resolved. Lemma 2 asserts that, for every irrational α, the spectrum of the corresponding values η_0 is nonempty, but its structure is unclear. Evidently, $\psi(t)$ does not correspond to an i. d. law.

Lemma 3. Let $G(u)$ be a continuous function nondecreasing on the segment $[\beta, 1]$ $(\beta < 1)$, and let $G(1) - G(\beta) > 0$. Construct a random variable X with ch. f.

$$\varphi(t) = E\exp(itX) = \exp\left(-\gamma t^2 + \int_\beta^1 (e^{itu} - 1)\,dG(u)\right).$$

For sufficiently small $\nu > 0$ and a suitably chosen $\eta_0 > 0$, the function

$$\psi(t) = \varphi(t)\exp(-\nu(e^{\eta_0 it} - 1))$$

will be the ch. f. of some random variable Y (of course, not i. d.).

The proof of these lemmas is carried out by means of the sieve method and elements of the theory of elliptic modular functions. Simple in idea, it is rather cumbersome in execution.

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Note: Figure translations are in progress. See original paper for figures.

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