# Mathematics

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1957

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**Abstract**

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*Mathematics*

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# ON ONE METHOD FOR THE APPROXIMATE SOLUTION OF THE LAPLACE EQUATION

*(Presented by Academician S. L. Sobolev on 15 XII 1956)*

Let there be given to us, in a finite part of the $xy$-plane, a domain $G$, whose boundary $\Gamma$ consists of a finite number of rectifiable curves, and suppose it is required to solve in this domain the Laplace equation $\Delta u = 0$ with the Dirichlet boundary condition $u|_\Gamma = \varphi$. In the present work a method of approximate solution will be indicated which, as the accuracy of the result increases, requires a smaller increase in memory and in the number of arithmetical operations in comparison with known methods for solving the Laplace equation by means of finite-difference equations (see, for example, ([1])).

As usual ([2]), we form a system of finite-difference equations with respect to the values of the solution at the points $(ih, jh)$, which we call nodes $(i, j)$. The nodes $(i, j)$ and $(i', j')$ are called neighboring if

$$|i - i'| + |j - j'| = 1.$$

Let $\Pi$ be the set of interior nodes such that all the interior points of the segments joining the node with neighboring nodes lie inside $G$, and let $\Pi_0$ be the set of the remaining interior nodes.

At the nodes of the set $\Pi$ we put

$$\frac{\widetilde{\Delta} u_{ij}}{h^2} = \frac{u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{ij}}{h^2} = 0.$$

At the nodes of the set $\Pi_0$ we replace $\partial^2 u/\partial x^2$ and $\partial^2 u/\partial y^2$ by divided differences in the values $u_{ij}$ at the given node and at the boundary points nearest to $(i, j)$ and nodes lying on the segments joining the node $(i, j)$ with neighboring nodes. Let

$$l_{ij}(u_{ij}) = 0 \tag{1}$$

be the system of equations just enumerated.

Denote by $s$ the greatest number such that, for certain integers $m$ and $n$, all nodes $(i, j)$ satisfying the condition

$$|i - m \cdot 2^{s-1}|,\ |j - n \cdot 2^{s-1}| < 2^{s-1}$$

Fig. 1

Figure 1: Fig. 1

belong to $\Pi$.

Let $\Omega_p^s$ $(p = 1, 2, \dots, \alpha_s)$ be all possible squares, defined by the inequalities

$$(m_{ps} - 1) \cdot 2^{s-1}h \leq x \leq (m_{ps} + 1) \cdot 2^{s-1}h,$$

$$(n_{ps} - 1) \cdot 2^{s-1}h \leq y \leq (n_{ps} + 1) \cdot 2^{s-1}h,$$

all interior nodes of which belong to $\Pi$. Here $m_{ps}$ and $n_{ps}$ are certain integers.

Denote by $Q_p^s$ the set of interior nodes of the square $\Omega_p^s$ satisfying the conditions:

A. $(i - m_{ps} \cdot 2^{s-1})(j - n_{ps} \cdot 2^{s-1}) = 0$.

B. The node $(i, j)$ is removed by a distance not greater than $2^{s-2}h$ from the center of the square or from one of its sides; all nodes of this side, with the possible exception of the vertices of the square, belong to $\Gamma$.

Let $L_p^s = Q_p^s - \bigcup_{i=1}^{p-1} Q_i^s$. Put $L_s = \bigcup_{i=1}^{\alpha_s} L_i^s$ and $\Gamma_s = \Gamma + L_s$. In the same way we consider all possible squares $\Omega_p^{s-1}$, except those for which all side nodes, with the possible exception of the vertices of the square, or all interior nodes of the square lying on one of the straight lines $x = m_{p,s-1} \cdot 2^{s-2}h,\ y = n_{p,s-1} \cdot 2^{s-2}h$, belong to $\Gamma_s$. Similarly to $L_p^s$ and $L_s$, from the nodes not belonging to $\Gamma_s$ of the corresponding squares we form the sets $L_p^{s-1}$ and $L_{s-1}$, using in item B the set $\Gamma_s$ instead of $\Gamma$. Put $\Gamma_{s-1} = \Gamma_s + L_{s-1}$, and so on. We continue this process down to $s = 1$.

The values $u_{ij}$ at the nodes belonging to $L_p^k$ $(k = 1, 2, \dots, s;\ p = 1, 2, \dots, \alpha_k)$ are expressed, with the aid of the Green function of the Dirichlet problem for the square, in terms of the values $u_{ij}$ on the boundary of $\Omega_p^k$. Adding to these equations the equations of system (1) referring to the points of the set $L_0$, and solving with respect to the corresponding values $u_{ij}$, we obtain the system

*Fig. 1*

$$\tilde{u}_{ij} = \widetilde{A} u_{ij} + \psi. \tag{2}$$

Here by $\tilde{u}_{ij}$ we mean the vector of the values $u_{ij}$ on the set $\Gamma_0 - \Gamma$. It is obvious that all elements of the matrix $A$ are nonnegative and that the sums of the elements of the matrix $A$ over the rows do not exceed 1.

Let $N$ be the number of nodes of the set $L$. It is clear that, for $k < s$, any square having a common center with some one of the squares $\Omega_p^k$ and sides equal to $3 \cdot 2^k h$, parallel to the coordinate axes, has common points with the boundary $\Gamma$. On the basis of this remark and of the rectifiability of $\Gamma$, we conclude that for large $N$ the number of elements of the matrix $A$ different from 0 is $\sim N$ (3), while the number of equations of system (2), equal to the number of unknowns, is $\sim \sqrt{N}\ln N$.

Let us carry out several auxiliary estimates. Let $\omega_{ij}$ be the solution of the system

$$\widetilde{\Delta}\omega_{ij} = 0 \qquad \text{for } 0 < i, j < n = 2m;$$

$$\omega_{ij} = 0 \qquad \text{for } i = n,\ 0 \le j \le n; \quad 0 \le i \le n,\ j = 0, n;$$

$$\omega_{0j} + \varphi_j \ge 0 \qquad \text{for } 0 < j < n$$

and let $v_{ij}$ be the solution of the system

$$\widetilde{\Delta}v_{ij} + \frac{\lambda}{n^2}v_{ij} = 0 \qquad \text{for } 0 < i, j < n$$

with the same boundary conditions.

The following is true: for $\lambda < 8$ there exists a $c$, independent of $\lambda$, $n$, and the function $\varphi_j$, such that

$$0 \le \omega_{ij} \le v_{ij}e^{-c\lambda} \quad \text{for } m/2 \le i \le 2m,\ j = m \quad \text{and for } i = m,\ 0 \le j \le 2m. \tag{3}$$

Next, denote by $W_{ij}^r$ ($r$ an integer) the function determined by the conditions:

$$W_{ij}^r\big|_\Gamma = 0;$$

$$l_{ij}\left(W_{ij}^r\right) = \begin{cases} -1 & \text{for } \rho\big((ih, jh), \Gamma\big) \le rh; \\ 0 & \text{for the remaining } i \text{ and } j. \end{cases}$$

Let $D$ and $d$ be, respectively, the largest and smallest of the diameters of the curves making up $\Gamma$. We shall show that there exists a number $M(d/D)$ such that

$$W_{ij}^r \le Mr^2h^2. \tag{4}$$

It is obvious that $\rho\big((i_0h, j_0h), \Gamma\big) \le rh$, if $W_{i_0j_0}^r = \max_{ij} W_{ij}^r$.

Let $P_\alpha$ ($\alpha = 1, 2$) be the squares defined by the inequalities $|x - i_0 h| \le \alpha r h$, $|y - j_0 h| \le \alpha r h$, and let $S_\alpha$ be their boundaries. It is obvious that in $G \cap P_2$

$$W_{ij}^r \le Q_{ij}^r + V_{ij}^r,$$

where $Q_{ij}^r = \frac{1}{2}\left[(2rh)^2 - (i - i_0)^2 h^2\right]$, while $V_{ij}^r$ is determined as follows:

$$\begin{aligned} l_{ij}\left(V_{ij}^r\right) &= 0 && \text{at the interior nodes of } G \cap P_2; \\ V_{ij}^r &= W_{i_0 j_0}^r && \text{on } S_2 \cap G; \\ V_{ij}^r &= 0 && \text{on } \Gamma \cap (P_2 - S_2). \end{aligned}$$

For $rh < d/3\sqrt{2}$ there exists an arc of the boundary $\Gamma$ connecting some points of the contours $S_1$ and $S_2$ and lying between these contours. Let, for example, the point $((i_0 - r)h, y_0)$ be the end of this arc belonging to $S_1$. Put

$$q_{ij}^r = 1 - V_{ij}^r / W_{i_0 j_0}^r.$$

In the domain of definition of the function $q_{ij}^r + q_{i,\,2j_0 - j}^r$ one has

$$q_{ij}^r + q_{i,\,2j_0 - j}^r \ge \theta_{ij}^r,$$

where $\theta_{ij}^r$ is determined from the system:

$$\theta_{ij}^r = 1 \quad \text{for } i_0 - 2r < i < i_0 - r, \quad j = j_0;$$

$$\theta_{ij}^r = 0 \quad \text{for } i = i_0 - r, \quad j_0 - r \le j \le j_0 + r \quad \text{and for } |i - i_0|,\ |j - j_0| = 2r;$$

$$\widetilde{\Delta}\theta_{ij}^r = 0 \quad \text{for the remaining } i \text{ and } j \text{ such that } |i - i_0|,\ |j - j_0| < 2r.$$

It is not difficult to verify that for $r \ge 2$ there exists $\sigma > 0$, independent of $r$, such that

$$2q_{i_0 j_0}^r \ge \theta_{i_0 j_0}^r \ge \sigma, \qquad q_{i_0 j_0}^1 \ge \frac{1}{16}.$$

Now the proof of inequality (4) becomes obvious. To estimate the rate of convergence when solving system (2) by the method of successive approximations, we construct a majorant. Put

$$\Phi_{ij} = \min\left\{\frac{1}{h^2},\ \frac{1}{\rho^2((ih, jh), \Gamma)}\right\}.$$

Let $z_{ij}\big|_\Gamma = 0$ and $l_{ij}(z_{ij}) = -\gamma\Phi_{ij}$. With the aid of (4) it is proved that

$$z_{ij} \leq \gamma MB \ln N,$$

where $B$ is an absolute constant. Put $\gamma = \dfrac{1}{BM \ln N}$.

It is not difficult to show, relying on the definition of $z_{ij}$ and (3), that for any coordinate $(\tilde{z}_{ij})_k$ of the vector $\tilde{z}_{ij}$ the following holds:

$$(A\tilde{z}_{ij})_k \leq (\tilde{z}_{ij})_k, \qquad (A^2\tilde{z}_{ij})_k \leq e^{-\varkappa/\ln N}(\tilde{z}_{ij})_k,$$

where

$$\varkappa = \min \frac{c}{8BM},\ \frac{1}{6BM}.$$

At the nodes of the set $L$,

$$z_{ij} \geq \eta/16.$$

From the preceding inequalities it follows that for any $r_{ij}$,

$$\left\|A^{2\mu+1}\tilde{r}_{ij}\right\|_C \leq \min\left(1, 16\, BM \ln N e^{-\varkappa\mu/\ln N}\right)\left\|\tilde{r}_{ij}\right\|_C.$$

Hence we conclude:

1) $\left\|(E - A)^{-1}\right\| \leq T\left(\dfrac{d}{D}\right) \ln N \ln\ln N$;
2) in order to find the solution of system (2) with accuracy $\asymp 1/N$, it is sufficient to carry out $\asymp \ln^2 N$ iterations according to the formula

$$\tilde{u}_{ij}^{\mu+1} = A\tilde{u}_{ij}^{\mu} + \psi,$$

computing $\tilde{u}_{ij}^{\mu+1}$ with accuracy

$$\asymp \frac{1}{N \ln N \ln\ln N}.$$

For computing the elements of the matrix $A$, defined by means of the values of the functions sin and sh, $\asymp N\ln^2 N$ operations are sufficient, and consequently, for determining $\tilde{u}_{ij}$, $\asymp N\ln^4 N$ operations are sufficient. The memory used amounts to $\asymp \sqrt{N}\ln N$ numbers. Knowing $\tilde{u}_{ij}$, it is easy to find the values $u_{ij}$ at the required points (4).

Let the desired solution $u(x,y) \in H(p,M,\lambda)$, where $p+\lambda \leq 4$ (5). We have[6]

$$|u^{\text{IV}}_{xxxx}| + |u^{\text{IV}}_{yyyy}| \leq \frac{c}{[\rho((x,y),\Gamma)]^{4-p-\lambda}}. \tag{5}$$

Using (4) and (5) and carrying out some additional estimates, we obtain

$$\begin{aligned} |u(ih,jh) - u_{ij}| &\leq L_{p+\lambda} h^{p+\lambda} && \text{for } p+\lambda < 2; \\ |u(ih,jh) - u_{ij}| &\leq L_2 h^2 |\ln h| && \text{for } p+\lambda = 2; \\ |u(ih,jh) - u_{ij}| &\leq L_{p+\lambda} h^2 && \text{for } p+\lambda > 2. \end{aligned}$$

These estimates are an improvement of the estimates[6] and, together with (5), carry over without essential changes to the multidimensional case under the following assumption: there exists a cone of finite height whose vertex can be placed at any boundary point in such a way that the cone and the domain have no common interior points.

The author expresses gratitude to R. Z. Khas' minskii, who suggested to the author the idea of proving inequality (4).

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Received
7 XII 1956

## REFERENCES

1. D. Young, *Trans. Am. Math. Soc.*, **76**, No. 1 (1954).
2. V. E. Miln, *Numerical Solution of Differential Equations*, IL, 1955.
3. A. N. Kolmogorov, *DAN*, **108**, 3 (1956).
4. N. S. Bakhvalov, *DAN*, **113**, No. 2 (1957).
5. N. M. Günter, *Potential Theory and Its Application to Basic Problems of Mathematical Physics*, 1953.
6. E. A. Volkov, *DAN*, **96**, 5 (1951).