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A. A. VASHARIN

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Abstract

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MATHEMATICS

A. A. VASHARIN

BOUNDARY PROPERTIES OF FUNCTIONS HAVING A FINITE DIRICHLET INTEGRAL WITH WEIGHT

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1. Let a domain $\Omega(x, y)$ of two variables be given, bounded by a sufficiently smooth boundary Γ. On Ω a positive function $\sigma(x, y)$ is defined, $2k$ times continuously differentiable, subject to the condition

$$c_1 \rho(x, y) \leq \sigma(x, y) \leq c_2 \rho(x, y),$$

where $\rho(x, y)$ is the distance of the point (x, y) to the boundary Γ along the normal, and c_1, c_2 are positive constants independent of x, y.

We shall say that a function $f(x, y) \in W^k_{2(\alpha)}$ if it has in Ω generalized derivatives in the sense of S. L. Sobolev up to order k, and

$$D^k_\alpha(f) = \iint\limits_\Omega \left[\sum_{\beta_1+\beta_2=k} \frac{k!}{\beta_1!\,\beta_2!} \left(\frac{\partial^k f}{\partial x^{\beta_1} \partial y^{\beta_2}} \right)^2 \right] \sigma^\alpha \, dx \, dy < \infty, \tag{1}$$

where $0 \leq \alpha < 1$ $(k = 1, 2, ...)$. In what follows, by $f|_\Gamma = \varphi(s)$ we shall understand a function for which there is convergence

$$f|_{\Gamma_h} = \varphi(s) \to \varphi(s)$$

in the mean, where Γ_h is the boundary of the domain Ω_h, the totality of those points of the domain Ω whose distance to Γ is not less than h.

We shall consider, by definition, that a 2π-periodic function $\varphi(s)$ belongs to the class A_l (l an integer) if it has absolutely continuous derivatives up to order $l - 1$, and if the derivative of order l belongs to L_2 and satisfies the condition

$$I_h[\varphi^l(s)] = \int_0^\delta \int_0^{2\pi} \frac{\left|\varphi^l(s+h) - \varphi^l(s)\right|^2}{h^{2-\alpha}} \, ds \, dh < \infty,$$

where $\delta > 0$ is arbitrary. The classes A_l were considered by P. L. Ul' yanov ([6]), and for $\alpha = 0$ by V. M. Babich and L. N. Slobodetskii ([7]). The present note is devoted to establishing necessary and sufficient conditions for finiteness of the integral (1). Our results generalize the corresponding results of S. M. Nikol' skii ([3]), V. M. Babich, L. N. Slobodetskii ([7]), and L. D. Kudryavtsev ([4]).

Theorem 1. A. If $f \in W^k_{2(\alpha)}$, then the functions

$$\left.\frac{\partial^\lambda f}{\partial n^\lambda}\right|_\Gamma = \varphi_\lambda(s) \qquad (\lambda = 0, 1, \dots, k-1) \tag{2}$$

belong respectively to the classes $A^\alpha_{k-\lambda-1}$ (n is the interior normal to Γ).

B. Conversely, if on Γ there is given a system of functions $\varphi_\lambda(s)$ $(\lambda = 0, 1, \dots, \dots, k-1)$, belonging respectively to the classes $A^\alpha_{k-\lambda-1}$, then in Ω one can construct a function $f(x, y) \in W^k_{2(\alpha)}$ satisfying conditions (2).

For $\alpha = 0$ this is a result of V. M. Babich and L. N. Slobodetskii ([7]); for $\alpha > 0$ this result somewhat strengthens the corresponding result of L. D. Kudryavtsev ([4]).

2. We shall briefly present the course of the proof. Let a function $f(x, y) \in W^k_{2(\alpha)}$ be given. Then it is not difficult to establish that $\partial^{k-1} f / \partial n^{k-1}|_\Gamma = \varphi_{k-1}(s) \in L_2$. Applying then the embedding theorems of S. L. Sobolev, one can assert that the functions $\varphi_\lambda(s)$ $(\lambda = 0, 1, \dots, k-1)$ in equalities (2) have meaning.

Consider the set $W^k_{2(\alpha)}(f)$ of functions $\{\psi(x, y)\}$ satisfying equalities (2) and belonging to $W^k_{2(\alpha)}$. It is nonempty, since it contains $f(x, y)$. Put

$$\inf_{\psi \in W^k_{2(\alpha)}(f)} D^k_\alpha(\psi) = d \geq 0,$$

and let $\{\psi_m(x, y)\}$ be a minimizing sequence for which

$$\lim_{m \to \infty} D^k_\alpha(\psi_m) = d.$$

Lemma. If $\psi \in W^k_{2(\alpha)}(f)$, then

$$\left\| \frac{\partial^l \psi}{\partial x^\beta \partial y^{l-\beta}} \right\|^2_{L_2(\Omega)} \leq C \left\{ D^k_\alpha(\psi) + \sum_{j=1}^{k-1} \left\| \frac{\partial^j \psi}{\partial n^j} \right\|^2_{L_2(\Gamma)} \right\}, \tag{3}$$

where C is a constant independent of $\psi(x, y)$, $l = 0, 1, \dots, k-1$.

On the basis of this lemma it is established in the usual way that the functions $\psi_m(x, y)$ of the minimizing sequence converge in the sense of $L_2(\Omega)$, together with their partial derivatives up to order k inclusive, to some function $u(x,y) \in W^k_{2(\alpha)}(f)$, and $D^k_\alpha(u) = d$. Hence it follows that the limiting function $u(x,y)$ is a generalized solution of the equation

$$L(u) = \sum_{\beta_1+\beta_2=k} \frac{k!}{\beta_1!\beta_2!} \frac{\partial^k}{\partial x^{\beta_1} \partial y^{\beta_2}} \left(\sigma^\alpha \frac{\partial^k u}{\partial x^{\beta_1} \partial y^{\beta_2}} \right) = 0 \tag{4}$$

under the boundary condition (2). Following the general scheme of S. L. Sobolev ([1]), we further establish that the limiting function $u(x,y)$ is $2k$ times continuously differentiable inside the domain Ω, and thus is an ordinary solution of the equation, and moreover unique in the class $W^k_{2(\alpha)}$.

Let now in the integral (1) $k=1$, Ω be the unit disk with center at the origin, and $\sigma(x,y) \equiv 1-\rho(x,y)$, where $\rho(x,y) \equiv (x^2+y^2)^{1/2}$. In this case the function $u(x,y)$ solving the variational problem satisfies the equation

$$\frac{\partial}{\partial \rho} \left[\rho(1-\rho)^\alpha \frac{\partial u}{\partial \rho} \right] + \frac{(1-\rho)^\alpha}{\rho} \frac{\partial^2 u}{\partial \theta^2} = 0 \tag{5}$$

and the boundary condition

$$u(1,\theta) = f(1,\theta) = \varphi(\theta), \tag{6}$$

where the solution is unique and can be represented in the form

$$u(\rho,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[\left(\rho^n + \sum_{k=1}^{\infty} c_{kn} \rho^{k+n} \right) (a_n \cos n\theta + b_n \sin n\theta) \right],$$

where

$$c_{kn} = \prod_{j=1}^{k} \frac{[(j+n-1)(j+n-1+\alpha) - n^2]}{[(j+n)^2 - n^2]},$$

a_n and b_n are chosen from the boundary condition (6). For c_{kn} it is true that

$$c' n^{\alpha/2} \le \sum_{k=1}^{\infty} c_{kn} \le c'' n^{\alpha/2},$$

where c' and c'' do not depend on n.

It follows from this that

$$m_1 \sum_{n=1}^{\infty} n^{1-\alpha}(c_n^2 + d_n^2) \leq D_\alpha(u) \leq m_2 \sum_{n=1}^{\infty} n^{1-\alpha}(c_n^2 + d_n^2), \tag{7}$$

where c_n and d_n are the Fourier coefficients of the function $\varphi(\theta)$; m_1 and m_2 are constants.

On the other hand, the inequality

$$k_1 I_h(\varphi) \leq \sum_{n=1}^{\infty} n^{1-\alpha}(c_n^2 + d_n^2) \leq k_2 I_h(\varphi), \tag{8}$$

holds, where k_1 and k_2 do not depend on n. Therefore, from conditions (1), (7), (8) it follows that $\varphi(\theta) \in A_0^\alpha$. Thus the theorem for the disk is proved.

If now a function $f(x, y) \in W^1_{2(\alpha)}$ is given in $\Omega(x, y)$, where $\Omega(x, y)$ is an arbitrary domain bounded by a sufficiently smooth boundary, then a small part of this domain adjacent to a sufficiently small arc of the boundary Γ can, by means of a continuously differentiable transformation a sufficient number of times, be mapped onto a half-disk in such a way that the boundary values remain the same and the class to which the function belongs is preserved. Then this function can be continued, with preservation of the class, to the whole disk, and, applying what was proved for the disk, one obtains the conditions required by the theorem for the boundary function, so far for a sufficiently small but arbitrary arc Γ of the domain Ω. The passage to the whole of Γ is carried out trivially with the aid of the Heine–Borel theorem. The passage from $k = 1$ to an arbitrary k is carried out by induction. The sufficient condition of the theorem reduces to establishing the possibility of the required continuation of the boundary values to a sufficiently small neighborhood of an arbitrary point of the boundary Γ. Then the arguments are carried out as in S. M. Nikol' skii (see [2], p. 317).

From what has been set forth it follows:

Theorem 2. *Under the conditions of Theorem 1 imposed on the functions $\varphi_\lambda(s)$, there exists in the domain Ω, and moreover uniquely in the class $W^k_{2(\alpha)}$, a function $u(x, y)$ satisfying the differential equation (4) and the boundary conditions (2).*

Indeed, from conditions B of Theorem 1 there follows the existence of a function $f \in W^k_{2(\alpha)}$ admissible for the variational problem and the fulfillment of conditions (2). Next one should repeat the beginning of the proof of Theorem 1A.

Mathematical Institute named after V. A. Steklov
Academy of Sciences of the USSR

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Note: Figure translations are in progress. See original paper for figures.