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Abstract

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On the Integrability of the Equations of Motion of a Rigid Body about a Fixed Point under the Action of a Central Newtonian Force Field

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§ 1. In the present work we consider the problem indicated in the title, under the assumption that the fixed point of the body is at a sufficiently large distance R, in comparison with the dimensions of the body, from the center of attraction. Then the components of the Newtonian force of attraction along fixed coordinate axes associated with the fixed point of the body may be written, with accuracy up to small quantities of second order, in the form:

$$F_x = -\frac{g}{R}\,dm\,x, \qquad F_y = -\frac{g}{R}\,dm\,y, \qquad F_z = -g\,dm + 2\frac{g}{R}\,dm\,z. \tag{1}$$

Here g is the acceleration of gravity at a distance R from the attracting center; dm is the mass of the attracted particle of the body; x, y, z are its coordinates in the fixed axes, which are constructed so that the origin coincides with the fixed point, while the z-axis is directed from the attracting center along the straight line joining the attracting center and the fixed point of the body. The forces (1) are applied to every point of the body.

Introduce a moving coordinate system x', y', z', whose origin coincides with the origin of the fixed coordinate system, and whose axes are directed along the principal axes of inertia of the body. Let p, q, r be the projections of the angular velocity of the body on the moving axes; $\gamma, \gamma', \gamma''$ the direction cosines of the moving axes with the z-axis; x'_0, y'_0, z'_0 the coordinates of the center of mass of the body in the moving axes.

Then the equations of motion of the rigid body under the action of the forces (1) have the form:

$$A\frac{dp}{dt} + (C-B)qr = -Mg\,(y'_0\gamma'' - z'_0\gamma') + 3\frac{g}{R}(C-B)\gamma'\gamma'';$$

$$B\frac{dq}{dt} + (A-C)rp = -Mg\,(z'_0\gamma - x'_0\gamma'') + 3\frac{g}{R}(A-C)\gamma''\gamma; \tag{2}$$

$$C\frac{dr}{dt} + (B-A)pq = -Mg\,(x_0'\gamma' - y_0'\gamma) + 3\frac{g}{R}(B-A)\gamma\gamma';$$

$$\frac{d\gamma}{dt} = r\gamma' - q\gamma''; \qquad \frac{d\gamma'}{dt} = p\gamma'' - r\gamma; \qquad \frac{d\gamma''}{dt} = q\gamma - p\gamma'.$$

Here A, B, C are the principal moments of inertia of the body at the fixed point, and M is the mass of the body.

Equations (2) generalize the equations of the classical problem of the motion about a fixed point of a heavy rigid body. Jacobi' s theory of the last multiplier is applicable to equations (2); as a consequence, the problem of integrating these equations is reduced to quadratures if four first independent algebraic integrals not containing time are known.

1. In the general case, the system of equations (2) admits three first independent integrals:

the energy integral

$$Ap^2 + Bq^2 + Cr^2 + 2Mg\,(x_0'\gamma + y_0'\gamma' + z_0'\gamma'') +$$

$$+3\frac{g}{R}\left(A\gamma^2 + B\gamma {a'}^2 + C\gamma''^2\right) = \text{const};$$

the integral of the angular momentum

$$Ap\gamma + Bq\gamma' + Cr\gamma'' = \text{const};$$

the relation between the direction cosines

$$\gamma^2 + \gamma {a'}^2 + \gamma''^2 = 1.$$

2. If the body has complete kinetic symmetry, i.e.

$$A = B = C,$$

then the system of equations (2) admits a fourth integral

$$x_0'p + y_0'q + z_0'r = \text{const}.$$

This case may be regarded as the degeneration of the following, more general, case.

3. Suppose the body has kinetic symmetry about some principal axis of inertia, and the center of mass lies on this axis. For example,

$$A = B, \qquad x'_0 = y'_0 = 0.$$

Then equations (2) admit the fourth integral

$$r = \text{const}$$

(an analogue of Lagrange' s case in the problem of the motion of a heavy rigid body).

4. If the body is fixed at its center of mass, i.e.

$$x'_0 = y'_0 = z'_0 = 0,$$

then equations (2) admit a fourth integral—the integral of kinetic momentum:

$$A^2p^2 + B^2q^2 + C^2r^2 - 3\frac{g}{R}\left(BC\gamma^2 + AC\gamma{a'}^2 + AB\gamma''^2\right) = \text{const}$$

an analogue of Euler' s case in the problem of the motion of a heavy rigid body).

§ 2. The presence in the equations of motion and in the first integrals of terms with the multiplier $3g/R$ qualitatively changes, in some cases, the picture of the motion in comparison with the motion, under the same conditions, of a heavy body.

Let us consider a particular solution of equations (2), possible for $y'_0 = 0$:

$$p = r = \gamma' = 0; \qquad \gamma'' = \gamma''(t); \qquad \gamma = \gamma(t); \qquad q = q(t).$$

This solution describes plane motion of the body (a generalization of the problem of the physical pendulum).

The system (2) reduces to one second-order equation

$$B\ddot{\vartheta} + Mga\sin(\vartheta + \delta) + 3\frac{g}{R}(A - C)\sin\vartheta\cos\vartheta = 0. \tag{3}$$

Here $\vartheta = \arccos\gamma''$; $a = \sqrt{x_0'^2 + y_0'^2}$; $\delta = \arcsin\frac{x'_0}{a} = \arccos\left(-\frac{z'_0}{a}\right)$ ...

Equation (3) reduces to a quadrature:

$$t - t_0 = \int_{\vartheta_0}^{\vartheta} \frac{d\vartheta}{\sqrt{l + \frac{2Mga}{B}\cos(\vartheta + \delta) + \frac{3}{2}\frac{g}{R}\frac{A-C}{B}\cos 2\vartheta}}, \tag{4}$$

where

$$l=\omega_0^2-\frac{2Mga}{B}\cos(\vartheta_0+\delta)-\frac{3}{2}\frac{g}{R}\frac{A-C}{B}\cos 2\vartheta_0;\quad \vartheta_0=\vartheta(t_0);\ \omega_0=\dot\vartheta(t_0).$$

Inversion of the integral (4) gives the solution of the problem.

Let the body be fixed at the center of mass, i.e. $a=0$. Then the integral (4) reduces to the elliptic integral

$$t-t_0=\frac{1}{\sqrt{3\frac{g}{R}\frac{A-C}{B}}}\int_{u_0}^{u}\frac{dy}{\sqrt{(1-u^2)(1-k^2u^2)}},\tag{5}$$

where

$$A>C;\quad u=\frac{1}{k}\sin\vartheta;\quad k^2=\frac{\omega_0^2}{3\frac{g}{R}\frac{A-C}{B}}+\sin^2\vartheta_0.$$

It follows from this that, in the absence of an initial angular velocity, a body fixed at the center of mass in a central force field will not be in indifferent equilibrium, as is the case in a plane-parallel homogeneous force field, but will execute periodic motion with period

$$T=\frac{2\pi}{\sqrt{3\frac{g}{R}\frac{A-C}{B}}}\left\{1+\left(\frac{1}{2}\right)^2\sin^2\vartheta_0+\left(\frac{1\cdot 3}{2\cdot 4}\right)^2\sin^4\vartheta_0+\cdots\right\}.$$

If $\omega_0\neq 0$, then the motion remains periodic under the condition $k^2<1$, i.e.

$$\omega_0<\sqrt{3\frac{g}{R}\frac{A-C}{B}}\ |\cos\vartheta_0|.$$

§ 3. In the present paragraph the Lagrange case is reduced to a quadrature: the body possesses kinetic symmetry ($A=B$), and the center of mass lies on the axis of symmetry ($x_0'=y_0'=0$).

The equations of motion (2) take the form:

$$\frac{dp}{dt}-mqr=-\alpha\gamma'-\beta m\gamma'\gamma'';\quad \frac{dq}{dt}+mpr=\alpha\gamma+\beta m\gamma''\gamma;\quad \frac{dr}{dt}=0,\tag{6}$$

where it is denoted

$$m = \frac{A - C}{A}; \quad \alpha = -\frac{Mgz_0'}{A}; \quad \beta = \frac{3g}{R}.$$

The integrals of the equations of motion have the form:

$$\begin{aligned} p^2 + q^2 - 2\alpha\gamma'' - m\beta\gamma''^2 &= h; \\ p\gamma + q\gamma' - (m-1)r\gamma'' &= k; \\ r &= r_0; \\ \gamma^2 + \gamma {a'}^2 + \gamma''^2 &= 1. \end{aligned} \tag{7}$$

For $\beta = 0$, from (6) and (7) we obtain the relations of the classical problem of the motion of a heavy rigid body in the Lagrange case.

From (6) we express γ and γ' in terms of the other variables, substitute these expressions into equations (7), and make the change of variables

$$p = \rho\cos\sigma, \qquad q = \rho\sin\sigma.$$

Then we obtain the equations

$$\rho^2\frac{d\sigma}{dt} + mr_0\rho^2 - (m-1)r_0\gamma''(\alpha + \beta m\gamma'') = k(\alpha + \beta m\gamma''); \tag{8}$$

$$\left(\frac{d\rho}{dt}\right)^2 + \rho^2\left(\frac{d\sigma}{dt}\right)^2 + 2mr_0\rho^2\frac{d\sigma}{dt} + m^2r_0^2\rho^2 + \gamma''^2(\alpha+\beta m\gamma'')^2 = (\alpha+\beta m\gamma'')^2; \tag{9}$$

$$\rho^2 - 2\alpha\gamma'' - \beta m\gamma''^2 = h. \tag{10}$$

After multiplication by $4\rho^2$, equation (9) takes the form

$$\left(\frac{d\rho^2}{dt}\right)^2 + 4\left[\rho^2\frac{d\sigma}{dt} + mr_0\rho^2\right]^2 + 4\rho^2(\alpha + \beta m\gamma'')^2(\gamma''^2 - 1) = 0. \tag{11}$$

Let us now express $\rho^2\dfrac{d\sigma}{dt} + mr_0\rho^2$ in terms of γ'' from (8), and ρ^2 in terms of γ'' from (10), and substitute in (11).

As a result we obtain the equation

$$\left(\frac{d\gamma''}{dt}\right)^2 = -\beta m\gamma''^4 - 2\alpha\gamma''^3 - a\gamma''^2 - b\gamma'' + c \equiv P(\gamma''), \tag{12}$$

where a, b, c are expressed in terms of $m, r_0, h, k, \beta, \alpha$.

Separating the variables in (12) and integrating, we obtain

$$t + \bar{c} = \int \frac{d\gamma''}{\sqrt{P(\gamma'')}}. \tag{13}$$

The problem has been reduced to a quadrature. Inversion of the elliptic integral (13) gives the time-dependence $\gamma'' = \gamma''(t)$, after which the quantities p, q, γ, γ' are determined as functions of t.

§ 4. In the case when the fixed point coincides with the center of mass ($x_0' = y_0' = z_0' = 0$), equations (2) coincide in form with the equations of the problem reduced to quadratures in paper (1).

In paper (2) the problem of the motion of a rigid body in a Newtonian force field is solved for the case $A = B,\ x_0' = y_0' = 0$, under particular assumptions on the form of the body and the distribution of density in the body.

It should be noted that the existence of integrals of equations (2) follows from the general theory of D. N. Goryachev (3).

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Note: Figure translations are in progress. See original paper for figures.