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Abstract

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PHYSICS

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THE RELATIVISTIC COLLISION INTEGRAL

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Let some particle α enter a medium which is, generally speaking, inhomogeneous and of variable density. The particles making up the medium move with different velocities. We shall assume that only binary collisions of the particle α with particles of the medium play a role, and that external forces, which in general may act on the particles, do not affect these collisions. We shall consider all particles to be relativistic, while the event space is Galilean.

It is required to find the probability $T^{-1}(P_0)\,d\tau$ that the particle α, starting at some instant of time t_0 from the point x_0, y_0, z_0 with momentum $m\mathbf{u}_0$, collides during the interval of proper time $d\tau$ with some particle of the medium; m is the rest mass of the particle α. The symbol P_0 denotes the aggregate $\{x_0, y_0, z_0, t_0, \mathbf{u}_0\}$.

Obviously,

$$T^{-1}(P_0) = \sum_{\beta} T^{-1}_{\alpha\beta}(P_0), \tag{1}$$

where $T^{-1}_{\alpha\beta}(P_0)$ determines the corresponding probability of collision of the particle α with a definite particle β belonging to the medium. The sum is taken over all particles of the medium. We have omitted the index α on the left-hand side of this equality. Let $A_\beta(P)\,dx\,dy\,dz\,du_1\,du_2\,du_3$ in some inertial frame of reference be the probability that at the instant of time t the particle β is in the volume $dx\,dy\,dz$ with center at the point x, y, z, and that its momentum lies in the limits $m_\beta u_k$, $m_\beta(u_k + du_k)$, $k = 1, 2, 3$; m_β is the rest mass of the particle β. We shall denote the radius of interaction of particles α and β by $r_{\alpha\beta}(\tilde{v})$, where $\tilde{v}$ is the magnitude of their relative velocity.

The problem of how $T^{-1}_{\alpha\beta}(P_0)$ is related to the quantities $A_\beta(P)$ and $r_{\alpha\beta}(\tilde{v})$ is a special case of the problem of the probability that the world line of a particle intersects a hypersurface in the space F of particle states. In general form this problem was posed in (^{1}). The state of a particle was called the aggregate P of its space-time position and velocity. In the interval between two successive collisions of the particle β with other particles of the medium, the motion of the particle β is represented by a curve in the space F. We shall call this curve the world line of the particle β in the indicated interval. As follows from (^{1}), the function $A_\beta(P)$ introduced above, considered on an arbitrary hypersurface S in F, is the probability density that the world line of the particle β, in one of the intervals between its collisions with other particles of the medium, intersects the hypersurface S.

Let us first find $T^{-1}_{\alpha\beta}(P_0)$ in prerelativistic mechanics. Obviously, a collision occurs as soon as the particle β intersects the half-sphere of radius $r_{\alpha\beta}(\tilde{v})$ with center at the position of the particle α, directed with its convex side along the vector of the relative velocity $\mathbf{v}_0 - \mathbf{v}$ of these particles.

(v_0 is the velocity of particle α, v is the velocity of particle β). The equations of this hemisphere are as follows:

$$\begin{aligned}
x &= x_0 + r_{\alpha\beta}(\tilde{v})\{\sin\tilde{\theta}\cos\widetilde{\varphi}\cos\theta - \cos\tilde{\theta}\cos\widetilde{\varphi}\sin\theta\cos\varphi + \sin\widetilde{\varphi}\sin\theta\sin\varphi\},\\
y &= y_0 + r_{\alpha\beta}(\tilde{v})\{\sin\tilde{\theta}\sin\widetilde{\varphi}\cos\theta - \cos\tilde{\theta}\sin\widetilde{\varphi}\sin\theta\cos\varphi - \cos\widetilde{\varphi}\sin\theta\sin\varphi\}, \qquad (2)\\
z &= z_0 + r_{\alpha\beta}(\tilde{v})\{\cos\tilde{\theta}\cos\theta + \sin\tilde{\theta}\sin\theta\cos\varphi\},
\end{aligned}$$

where

$$\tilde{v}\sin\tilde{\theta}\cos\widetilde{\varphi} = v_{01} - v_1; \qquad \tilde{v}\sin\tilde{\theta}\sin\widetilde{\varphi} = v_{02} - v_2; \qquad \tilde{v}\cos\tilde{\theta} = v_{03} - v_3;$$

$0 \le \theta \le \pi/2$; $0 \le \varphi < 2\pi$. These equations, together with the equation $t = t_0$, define in F a five-dimensional surface $\Gamma(P_0)$. Since we have assumed that external forces do not affect collisions of particles, when considering collisions one may suppose that there are no external forces. Therefore one may assume that, during an interval $d\tau$ of the proper time of particle α, its state P_0 is displaced in the space F by the infinitesimal vector $\{v_0\,d\tau, d\tau, 0, 0, 0\}^*$. At the same time every point of the surface $\Gamma(P_0)$ is displaced by the same vector, and this surface sweeps out a six-dimensional strip ΔS. The desired probability $T^{-1}_{\alpha\beta}(P_0)d\tau$ is equal to the probability that the world line of particle β intersects this strip. Hence we find:

$$T^{-1}_{\alpha\beta}(P_0) = \int_{-\infty}^{+\infty} r^2_{\alpha\beta}(\tilde{v})\,\tilde{v}\,(dv)\int_0^{\pi/2}\int_0^{2\pi} A_\beta(x, y, z, t_0, \mathbf{v})\cos\theta\sin\theta\,d\theta\,d\varphi, \qquad (3)$$

where x, y, z are determined by formulas (2). If the values of the function $A_\beta(P)$ on the hemisphere (2) differ little from its value at the center x_0, y_0, z_0 of the hemisphere, then $T^{-1}_{\alpha\beta}(P_0)$ takes the usual form:

$$T^{-1}_{\alpha\beta}(P_0) = \int_{-\infty}^{+\infty} A_\beta(x_0, y_0, z_0, t_0, \mathbf{v})\, \pi r^2_{\alpha\beta}(\tilde{v})\, \tilde{v}\, (dv). \tag{4}$$

Under this condition, by the same method one can also find $T^{-1}_{\alpha\beta}(P_0)$ in relativistic mechanics**. It is only necessary to replace the hemisphere by a two-dimensional element in the space of events, perpendicular to the vectors $\{\mathbf{u}, u_4\}$ and $\{\mathbf{u}_0, u_{04}\}$, with center at the point x_0, y_0, z_0, t_0 and with measure $\sigma_{\alpha\beta}(E_{a_0 a})$, equal to the cross section for the interaction of particles α and β, where

$$E_{a_0 a} = mc^2 \left\{ \operatorname{ch} \frac{S_{a_0 a}}{c} - 1 \right\},$$

$$\operatorname{ch} \frac{S_{a_0 a}}{c} = u_4 u_{04} - \frac{\mathbf{u}\mathbf{u}_0}{c^2}, \qquad u_4 = \sqrt{1 + u^2/c^2}.$$

If the state P_0 is fixed, then the totality of such elements fills in the space F a five-dimensional surface $\gamma(P_0)$. During an interval $d\tau$ of the proper time of particle α, its state P_0 is displaced in the space F by the infinitesimal vector $\{\mathbf{u}_0\, d\tau, u_4\, d\tau, 0, 0, 0\}$. At the same time every point of the surface $\gamma(P_0)$ is displaced by the same vector, and thus a six-dimensional strip is swept out. The desired probability $T^{-1}_{\alpha\beta}(P_0)d\tau$, as before, is equal to the probability that the world line of particle β intersects this strip. Hence

$$T^{-1}_{\alpha\beta}(P_0) = \int_{-\infty}^{+\infty} A_\beta(x_0, y_0, z_0, t_0, \mathbf{u})\, \sigma_{\alpha\beta}(E_{a_0 a})\, c \operatorname{sh} \frac{S_{a_0 a}}{c}\, da, \tag{5}$$

where

$$da = \frac{du_1\, du_2\, du_3}{u_4}.$$

Thus, the problem posed is solved.

Let us also suppose that the interaction time of particles α and β is small. Starting from the state P^*, the particle α during the interval of its proper time

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* Here and below, in relativistic mechanics, as coordinates in F we choose the Galilean coordinates x, y, z, t of the particle and the coordinates u_1, u_2, u_3 of its velocity, equal to the ratios of the spatial components of its momentum to the rest mass.

** We restrict ourselves to this condition because, in the case of an arbitrary interaction radius of particles α and β, the relativistic treatment apparently encounters difficulties connected with the kinematics of the particle system.

$d\tau$ may collide with particle β and then cease to interact with it, being in a state belonging to the elementary domain $d \subset F$ with center at the point P. Under the assumptions made above, the probability of such an event is equal to

$$\delta(x - x^*)\delta(y - y^*)\delta(z - z^*)\delta(t - t^*)\mathcal{L}_{\alpha\beta}(P; a^*)\mu(d)d\tau, \tag{6}$$

where $\mu(d) = dx\,dy\,dz\,dt\,da$. Let the differential scattering cross section of a particle α with velocity a^* on a particle β with velocity a' be equal to

$$d\sigma = H_{\alpha\beta}(E_{a^*a'}, E_{aa'}, \cos\theta\,)\,da, \tag{7}$$

where

$$\cos\theta \;= \frac{\operatorname{ch}\dfrac{S_{a^*a'}}{c}\operatorname{ch}\dfrac{S_{aa'}}{c} - \operatorname{ch}\dfrac{S_{aa^*}}{c}}{\operatorname{sh}\dfrac{S_{a^*a'}}{c}\operatorname{sh}\dfrac{S_{aa'}}{c}}. \tag{8}$$

Replacing in the derivation of formula (5) $\sigma_{\alpha\beta}(E_{a_0a})$ by $d\sigma$, we obtain that the function $\mathcal{L}_{\alpha\beta}$ entering (6) has the form:

$$\mathcal{L}_{\alpha\beta}(P; a^*) = \int_{-\infty}^{+\infty} A_\beta(x, y, z, t, u')H_{\alpha\beta}(E_{a^*a'}, E_{aa'}, \cos\theta\,)c\operatorname{sh}\frac{S_{a^*a'}}{c}\,da'. \tag{9}$$

On the basis of (1) and of the results obtained here one can write the relativistic kinetic equation of motion of particles α in matter. Let T_0^{-1} be the probability of spontaneous decay of a particle α per unit of its proper time. We denote

$$\mathcal{L}(P; a^*) = \sum_\beta \mathcal{L}_{\alpha\beta}(P; a^*), \tag{10}$$

where the sum, as in (1), is taken over all particles making up the matter. The equation named has the form:

$$u_4\frac{\partial A(P)}{\partial t} + \mathbf{u}\nabla A(P) + u_4\frac{\partial}{\partial \mathbf{u}}\left[\frac{\vec{\omega}(P)}{u_4}A(P)\right] + \left[T^{-1}(P) + T_0^{-1}\right]A(P) = \lambda\int_{-\infty}^{+\infty} A(P^*)\mathcal{L}(P; a^*)\,da^* + Q(P). \tag{11}$$

We have omitted the index α. The expansion of the function $A(P; \lambda)$ in a series in powers of λ in the present case means an expansion in the number of collisions of the particle α in the matter (for the remaining notation see (^{1})).

The quantity

$$\int_{-\infty}^{+\infty} A(P^*)\mathcal{L}(P;a^*)\,da^* - A(P)T^{-1}(P) \tag{12}$$

is the relativistic Boltzmann collision integral, in which the collisions of the particle α with the particles of the matter are taken into account.

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CITED LITERATURE

(1) N. A. Chernikov, DAN, **112**, No. 6 (1957).

Note: Figure translations are in progress. See original paper for figures.