Mathematics

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Abstract

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DEGENERATION IN THE TRANSITION FROM A DISCRETE SPECTRUM TO A CONTINUOUS ONE AND THE TRANSITION FROM QUANTUM MECHANICS TO CLASSICAL MECHANICS

(Presented by Academician N. N. Bogolyubov, 31 X 1956)

Let $\{e_i^k\}$ be a system $(k = 1, 2, ...)$ of orthonormal bases in some Hilbert space H; let $\{n_k\}$ be a certain sequence of integers depending on k, $\lim_{k\to\infty} n_k = \infty$. In what follows, in $\{n_k\}$ we shall omit the index k.

Definition 1. An element of the space $M_n^0[e_i^k]$ is a sequence $\{x_k\}$, where

$$x_k = \sum_{i=-n}^{\infty} a_i e_{i+n}^k \in H, \qquad \sum_{i=-\infty}^{\infty} a_i^2 < \infty. \tag{1}$$

Definition 2. An element of the space $M_n[e_i^k]$ is a class of equivalent sequences in H of the form

$$\sum_{i=-n}^{\infty} a_i e_{i+n}^k + \sum_{i=1}^{\infty} o_{ik} e_i^k,$$

where o_{ik} satisfy the condition

$$\lim_{k\to\infty} \sum_{i=1}^{\infty} o_{ik}^2 = 0. \tag{2}$$

Sequences of the form $\sum_{i=1}^{\infty} o_{ik} e_i^k$ satisfying (2) are taken as elements equivalent to zero.

Define in $M_n[e_i^k]$ the scalar product as

$$\lim_{k\to\infty} (x_k, \bar{x}_k), \qquad x_k, \bar{x}_k \in M_n[e_i^k]. \tag{3}$$

If

$$x_k = \sum_{i=-n}^{\infty} [a_i + o_{ik}] e^k_{i+n}, \qquad \bar{x}_k = \sum_{i=-n}^{\infty} [\bar{a} + \bar{o}_{ik}] e^k_{i+n},$$

then, by virtue of (2),

$$\lim_{k\to\infty} (x_k, \bar{x}_k) = \sum_{i=-\infty}^{\infty} a_i \cdot \bar{a}_i.$$

The space $M_n[e^k_i]$ is a separable Hilbert space, since the elements of $M_n[e^k_i]$ are put into one-to-one correspondence with sequences $\{a_i\}$, $i = 1, -1, ...$, from $l_2 \oplus l_2$. A countable basis in this latter space is formed by the elements $\{\delta^j_i\}$, $i = 1, -1, ...$; $\delta^j_i = 1$ for $i = j$; $\delta^j_i = 0$ for $i \neq j$. Hence it follows that a countable basis in the space $M_n[e^k_i]$ is formed by the elements $\{e^k_{n+i}\}$, $i = 1, -1, ...$

If $(n - n') \to \infty$ as $k \to \infty$, then the spaces $M_n[e^k_i]$ and $M_{n'}[e^k_i]$ will be orthogonal in the sense of the scalar product

$$\lim_{k\to\infty} (x_k, \bar{x}_k): \quad x_k \in M_n[e^k_i], \quad \bar{x}_k \in M_{n'}[e^k_i], \tag{3a}$$

since the scalar product (in H) $(e^k_{n+i}, e^k_{n'+j})$ will be equal to zero for any i and j when k is sufficiently large.

Lemma. Let

$$x_k = \sum_{i=0}^{\infty} a_{ki} e^k_i \in H, \quad \lim_{k\to\infty} (x_k, x_k) = \sum_{i=-\infty}^{\infty} \lim_{k\to\infty} a^2_{k,n+i} < \infty.$$

Then $\{x_k\} \in M_n[e^k_i]$.

Let a sequence of operators $\{A_k\}$, acting in H, be given. We shall say that $x_k \in M^0_n[e^k_i]$ belongs to the domain of definition of the operator $\{A_k\}$ in the space $M^0_n[e^k_i]$ if, for each k, x_k belongs to the domain of definition of each operator A_k in H, and the sequence $\{A_k x_k\}$ belongs to $M_n[e^k_i]$. (Taking the corresponding element in $M^0_n[e^k_i]$, we define the action of the operator in $M^0_n[e^k_i]$, and thereby also in $M_n[e^k_i]$.)

Let $\{A_k\}$ be a sequence of self-adjoint positive-definite operators with discrete spectrum, with eigenfunctions $\{x^k_i\}$ and eigenvalues $\{\lambda^k_i\}$, converging strongly ⁽¹⁾ to the self-adjoint operator A with continuous spectrum. Let λ be a point of the spectrum of the operator A; n determines the number of the eigenvalue of the operator A_k nearest from above to λ. Denote by $M_\lambda\{A_k\}$ the space $M_n[x^k_i]$*.

Theorem 1. *The closure of the operator $\{A_k\}$ in $M_\lambda\{A_k\}$ is the operator of multiplication by the number λ.*

Proof. On the set $\{x^k_{n+i}\}$, $i = 1, -1, ...$, the operator $\{A_k\}$ is equivalent to the operator of multiplication by λ, since

$$A_k x^k_{n+i} = \lambda^k_{n+i} x^k_{n+i} = (\lambda + o_{ik}) x^k_{n+i},$$

where

$$o_{ik} = (\lambda^k_n - \lambda) + (\lambda^k_{n+i} - \lambda^k_n) \xrightarrow[k\to\infty]{} 0.$$

The set of elements $\{x^k_{n+i}\}$, $i = 1, -1, \ldots, -r, r, \ldots$, forms a countable basis in $M_\lambda\{A_k\}$; therefore, the closure of the operator $\{A_k\}$ in $M_\lambda\{A_k\}$ is the operator of multiplication by λ, as was required to prove.

In what follows we shall consider operators A_k whose eigenvalues satisfy the condition

$$(\lambda^k_{n+i} - \lambda)(\lambda^k_{n+s} - \lambda)^{-1} s^{1/2+\delta} \to 0 \tag{4}$$

as $s \to \infty$, uniformly in k as $k \to \infty$, for some $\delta > 0$ and any integer i.

Theorem 2. *Let $\{x_k\}$ be a sequence of elements of H, belonging to the domain of definition of the operators A_k and normalized to unity, $\|x_k\| = 1$.*

Suppose that the sequence

$$\{\|(A_k - \lambda)x_k\|(\lambda^k_{n+l} - \lambda)^{-1}\}$$

is uniformly bounded as $k \to \infty$ for some integer l. Then there exists a subsequence $\{m\}$ of the sequence $\{k\}$ such that $\{x_m\} \in M_\lambda\{A_m\}$.

* The spaces $M_{\lambda_1}\{A_k\}$ and $M_{\lambda_2}\{A_k\}$ for $\lambda_1 \neq \lambda_2$ are orthogonal in the sense of the definition of the scalar product (3a). Therefore one can form the space $M\{A_k\}$, equal to the continuous direct sum of the spaces $M_\lambda\{A_k\}$ with respect to the measure λ:

$$M\{A_k\} = \int_0^\infty \oplus M_\lambda\{A_k\}\, d\lambda\ ^{(2)}.$$

Proof. Let $x_k = \sum_{i=0}^\infty a_{ki} x^k_i$. Consider the sequence $\{a_{kn}\}$. It is bounded. Choose a convergent subsequence $\{a_{k'n}\}$. Consider the sequence $\{a_{k',n+1}\}$ and choose from it a convergent subsequence $\{a_{k'',n+1}\}$; then consider the sequence

$\{a_{k'', n-1}\}$, choose a convergent subsequence $\{a_{k''', n-1}\}$, etc. We obtain a collection of subsequences $\{k'\}, \{k''\}, \{k'''\}, \ldots$. From these choose the diagonal sequence: from $\{k'\}$ take the first term, from $\{k''\}$ the second, and so on. We obtain a subsequence $\{m\}$ such that the limit of $\{a_{m, n+i}\}$ as $m \to \infty$ exists for any i, $-\infty < i < \infty$; denote

$$\lim_{m \to \infty} a_{m, n+i} = a_i.$$

Let us estimate the coefficient $a_{m, n+i}$. For sufficiently large i, by (4) we obtain:

$$a_{m, n+i} = (x^m_{n+i}, x_m) = (\lambda^m_{n+i} - \lambda)^{-1}([A_m - \lambda] x^m_{n+i}, x_m) \leq$$

$$\leq (\lambda^m_{n+i} - \lambda)^{-1} \|(A_m - \lambda) x_m\| < \mathrm{const} \cdot (\lambda^m_{n+l} - \lambda)^{-1} (\lambda^m_{n+l} - \lambda) \leq i^{1/2 - \delta}.$$

Consequently, for any $\varepsilon > 0$ one can choose N such that, for all m greater than some M,

$$\sum_{i=-n}^{-N} a^2_{m, n+i} + \sum_{i=N}^{\infty} a^2_{m, n+i} < \varepsilon; \qquad \sum_{i=-\infty}^{-N} a_i^2 + \sum_{i=N}^{\infty} a_i^2 < \varepsilon.$$

For $m > M_\varepsilon > M$,

$$\sum_{i=-N}^{N} a^2_{m, n+i} - a_i^2 < \varepsilon.$$

Hence

$$\sum_{i=-n}^{\infty} a^2_{m, n+i} - \sum_{i=-\infty}^{\infty} a_i^2 < 2\varepsilon.$$

The condition of the lemma is fulfilled; consequently, $\{x_m\} \in M_\lambda \{A_m\}$. The theorem is proved.

Let $\{B_k\}$ be a sequence of operators acting in H; C_k is the commutator of A_k and B_k,

$$C_k = A_k B_k - B_k A_k.$$

Theorem 3. Suppose that the expressions

$$\|C_k x^k_{n+i}\| (\lambda^k_{n+l} - \lambda)^{-1}, \qquad \|B_k x^k_{n+i}\|$$

are uniformly bounded as $k \to \infty$ for all i and some l. Then there exists a subsequence of operators $\{B_m\}$, defined in the space $M_\lambda \{A_m\}$ as an operator on an everywhere dense set.

Proof. Since

$$[A_k - \lambda] B_k x^k_{n+i} = C_k x^k_{n+i} + B_k A_k x^k_{n+i} - \lambda B_k x^k_{n+i} = \{C_k + (\lambda^k_{n-i} + \lambda) B_k\} x^k_{n+i},$$

it follows that either

$$\|(A_k - \lambda) B_k x^k_{n+i}\| (\lambda^k_{n+i} - \lambda)^{-1},$$

or

$$\|(A_k - \lambda)B_k x^k_{n+i}\|(\lambda^k_{n+l} - \lambda)^{-1}$$

must be uniformly bounded as $k \to \infty$; consequently, by Theorem 2,

$$B_m x^m_{n+i} \in M_\lambda\{A_m\},$$

where $\{m\}$ is a subsequence of $\{k\}$.

From Theorems 2 and 3 follows the following assertion:

Let $f \in L_2$ be a bounded function, and suppose that the commutator $A_{kf} - fA_k$ satisfies the conditions of theorem (4). Suppose that the limits

$$\lim_{k\to\infty} (f x^k_{n+l}, x^k_{n+p})$$

exist for any p and l. Then the limits

$$\lim(f^r x^k_{n+l}, x^k_{n+p})$$

also exist for any integers l, p and $r > 0$.

Let A_k be equal to the three-dimensional Hamiltonian operator

$$H_k = -\frac{h^2}{2\mu}\frac{d^2}{dx^2} + u(x) \quad (u(x) \geq 0)$$

with a system of eigenfunctions $\{\psi_i\}$; let h depend on k so that $h_k \to 0$ as $k \to \infty$ (3). From the asymptotics of the eigenvalues (4) it follows that condition (3) will be fulfilled. The commutator of H with x is equal to

$$Hx - xH = -\frac{i\hbar}{\mu}p.$$

The sequence $\{p\psi_{n+i}\}$ is bounded ···

as $h \to 0$. Indeed,

$$p^2\psi_{n+i} + 2\mu u(x)\psi_{n+i} = 2\mu\lambda_{n+i}\psi_{n+i},$$

$$(\psi_{n+i}, p^2\psi_{n+i}) + 2\mu(\psi_{n+i}, u(x)\psi_{n+i}) = 2\mu\lambda_{n+i};$$

hence

$$(\psi_{n+i}, p^2\psi_{n+i}) = \|p\psi_{n+i}\|^2 \leq 2\mu\lambda_{n+i} \to 2\mu\lambda.$$

Thus the conditions of Theorem 3 are satisfied. Consequently, from the sequence $\{h_k\}$ one can choose a subsequence $\{h_m\}$ such that the operator of multiplication by x will be defined in $M_\lambda\{H_m\}$. In the one-dimensional case one can prove, proceeding from asymptotics, that the operator of multiplication by x is defined in the space $M_\lambda\{H_k\}$. It follows from Theorem 1 that, as $k \to \infty$, the Hamiltonian operator passes into the operator of multiplication by the total energy λ, i.e. into the corresponding physical quantity. The momentum operator $p = -i\frac{d}{dx}$ in the one-dimensional case passes into the operator of multiplication by $\sqrt{2\mu[\lambda - u(x)]}$ ($u(x)$ is the potential energy), i.e. into the classical momentum.

In the limit the following picture is obtained. In the space $M\{H_k\}$ there are defined the operators H, p, and the operator of multiplication by x. They commute with one another and are connected by the relation

$$H = \frac{p^2}{2\mu} + u(x).$$

The matrix element, corresponding to different energy levels λ_1 and λ_2, of any of the indicated operators is equal to zero. The mean value is equal to the classical mean value. However, in contrast to quantum mechanics, one can specify one more independent variable (x or p) and thereby determine the system completely.

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CITED LITERATURE

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2. M. A. Naimark, S. V. Fomin, *Uspekhi matem. nauk*, **10**, No. 2 (1955).
3. V. P. Maslov, DAN, **94**, No. 4 (1954); **111**, No. 3 (1956); **111**, No. 5 (1956).
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CORRECTION

In my article published in DAN, vol. 111, No. 3, 1956 (V. P. Maslov, "The theory of perturbations of linear operator equations and the problem of a small parameter in differential equations"), Theorem 5 should read:

Theorem 5. *The solution of the elliptic equation*

$$\varepsilon \Delta u - c^2(x_1, \dots, x_p)u = F(x_1, \dots, x_p, \varepsilon); \qquad F(x_1, \dots, x_p, \varepsilon) \in L_2$$

with zero boundary condition $u|_\Gamma = 0$ *converges in mean to*

$$u_0 = \frac{F(x_1, \dots, x_p, 0)}{c^2(x_1, \dots, x_p)},$$

if the square of the function u_0 *is integrable in the domain bounded by the hypersurface* Γ *(where* Γ *is assumed sufficiently smooth).*

Note: Figure translations are in progress. See original paper for figures.

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