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Abstract

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ON THE EXISTENCE OF SOLUTIONS OF THE CAUCHY PROBLEM FOR A CERTAIN CLASS OF PARTIAL DIFFERENTIAL EQUATIONS

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Until recently, the principal attention was directed to the study of classes of existence of solutions of the Cauchy problem for systems of linear equations of the form

$$\frac{\partial u(x,t)}{\partial t} = P\left(\frac{1}{i}\frac{\partial}{\partial x}\right)u(x,t). \tag{1}$$

The works of S. L. Sobolev ([1]) and S. A. Galpern ([2]) showed that the problem of finding classes of existence of solutions of the Cauchy problem for systems

$$\frac{\partial}{\partial t}P_1\left(\frac{1}{i}\frac{\partial}{\partial x}\right)u(x,t) = P_2\left(\frac{1}{i}\frac{\partial}{\partial x}\right)u(x,t) \tag{2}$$

is of considerable interest.

In the present work one equation of the form (1) is considered in the one-dimensional space $-\infty < x < \infty$, with the initial condition

$$u(x,0) = u_0(x) \tag{2a}$$

and a class of existence of solutions of (2), (2a) is found. In doing this, the method of generalized functions, developed by I. M. Gel' fand and G. E. Shilov in ([3]), is applied.

Theorem. If the order of the differential polynomial $P_1\left(\frac{1}{i}\frac{\partial}{\partial x}\right)$ is greater than the order of $P_2\left(\frac{1}{i}\frac{\partial}{\partial x}\right)$, and if $u_0(x)$ satisfies the inequality

$$|u_0(x)| < A_1 \exp[-A_2|x|^{q/(q+1)}], \tag{3}$$

then for equation (2) there exists a solution of the Cauchy problem (2), (2a) in the class of functions

$$|u(x,t)| \le B_1 \exp[B_2|x|^{q/(q+1)}], \tag{3a}$$

where q is the largest multiplicity of a real root of the polynomial $P_1(s)$.

Let us note that here no assumptions are made concerning the differentiability of the initial functions. In this sense the equations considered by us resemble parabolic equations.

Proof. Following the scheme of I. M. Gel’ fand and G. E. Shilov, consider the equation

$$\frac{d}{dt}P_1(s)v(s,t) = P_2(s)v(s,t). \tag{4}$$

with the initial condition

$$v(s,0) = v_0(s). \tag{4a}$$

As the basic space we take the space R of infinitely differentiable functions $\varphi(\sigma)$ satisfying the inequalities

$$\left|\sigma^k \varphi^{(p)}(\sigma)\right| \le A_k B^p p^{\frac{q+1}{q}p} \exp\left[-\frac{A_1}{|\sigma - a_1|^{p_1+\varepsilon}}\right] \cdots \exp\left[-\frac{A_l}{|\sigma - a_l|^{p_l+\varepsilon}}\right], \tag{5}$$

where $a_1, a_2, \dots, a_l$ are the real roots of the polynomial $P_1(s)$; $p_1, p_2, \dots, p_l$ are their multiplicities.

In the corresponding class $T(R)$ of generalized functions the problem (4), (4a) has a solution, since $\exp[P_2(s)t/P_1(s)]$ is a multiplier (3) in the class of functions R.

Indeed, the fraction $P_2(s)/P_1(s)$ can be represented as the sum of two terms: $P_2(s)/P_1(s) = Q_2(s)/Q_1(s) + M_2(s)/M_1(s)$, such that $Q_1(s)$ has only real roots, while $M_1(s)$ has only non-real roots. Consequently,

$$\exp[P_2(s)t/P_1(s)] = \exp[Q_2(s)t/Q_1(s)] \exp[M_2(s)t/M_1(s)]. \tag{6}$$

The second factor in (6) is obviously a multiplier in R. We shall prove that the first factor is also a multiplier in R. Expanding the fraction $Q_2(s)/Q_1(s)$ into partial fractions, we obtain

$$\exp\left[\frac{Q_2(s)t}{Q_1(s)}\right]\prod_{j=1}^{l}\prod_{k=1}^{p_j}\exp\left[\frac{A_k^j t}{(s-a_j)^{p_j}}\right],$$

where $A_k^j = \psi_j^{(k-1)}(a_j)/(k-1)!$, $\psi_j(s) = Q_2(s)(s-a_j)^{p_j}/Q_1(s)$.

Here factors of the forms $\exp[A/(s-a)^k]$ and $\exp[Ai/(s-a)^k]$ may occur, where a and A are real numbers. By Cauchy' s formula,

$$D^m \exp\left[\frac{A}{(\sigma-a)^k}\right] = \frac{m!}{2\pi i}\int_\Gamma \frac{\exp\left[\frac{A}{(\xi+a)^k}\right]}{(\xi-\sigma)^{m+1}}\,d\xi, \tag{7}$$

where Γ is a circle tangent to two straight lines drawn through the point $(a,0)$ at an angle α to the x-axis (this angle depends on k), with its center at an arbitrary point σ of the real axis. Estimating (7) in modulus gives

$$\left|D^m \exp\left[\frac{A}{(\sigma-a)^k}\right]\right| \le B^m m^m \frac{\exp\left[\frac{A'}{|\sigma-a|^k}\right]}{|\sigma-a|^m}. \tag{8}$$

Taking (5) and (8) into account, we obtain

$$\left|\sigma^k D^p\left\{\exp\left[\frac{A}{(\sigma-a)^k}\right]\varphi(\sigma)\right\}\right| \le$$

$$\le A_k B^p p^{\frac{q+1}{q}p}\exp\left[-\frac{A_1'}{|\sigma-a_1|^{p_1+\varepsilon}}\right]\cdots\exp\left[-\frac{A_l'}{|\sigma-a_l|^{p_l+\varepsilon}}\right].$$

Thus, $D^p\left[\exp\left[\frac{A}{(\sigma-a)^k}\right]\varphi(\sigma)\right] \in R$, and therefore $\exp\left[\frac{A}{(\sigma-a)^k}\right]$ multiplier. For the second type of factors the proof can be carried out according to the same plan.

In order to prove the existence of a solution of the problem (2), (2a) in the class of ordinary functions, let us compute the Fourier transform of $\exp[P_2(s)t/P_1(s)]$. Since the function being transformed $\exp[P_2(s)t/P_1(s)]$ is the product of the two factors (6), we have ([5])

$$\exp\widetilde{\left[\frac{P_2(s)t}{P_1(s)}\right]} = \exp\widetilde{\left[\frac{Q_2(s)t}{Q_1(s)}\right]} * \exp\widetilde{\left[\frac{M_2(s)t}{M_1(s)}\right]}.$$

Let us find the Fourier transforms of each factor. The Fourier transform of the function $\exp[M_2(s)/M_1(s)]$ is the sum

$$\exp\left[\widetilde{\frac{M_2(s)}{M_1(s)}}\right] = \delta(x) + \alpha(x,t), \qquad \text{where } |\alpha(x,t)| < C_1 e^{-C_2|x|}. \tag{9}$$

The number C_2 is the minimum distance from the x-axis to the roots of the polynomial $M_1(s)$.

It is easy to show that the series

$$\exp\left[\frac{Q_2(s)t}{Q_1(s)}\right] = 1 + \sum_{n=1}^{\infty} \frac{1}{n!}\left(\frac{Q_2(s)t}{Q_1(s)}\right)^n \tag{10}$$

converges in the sense of generalized functions $T(R)$. Hence,

$$\exp\left[\widetilde{\frac{Q_2(s)t}{Q_1(s)}}\right] = \delta(x) + \varphi(x,t), \qquad \text{where } \varphi(x,t) = \sum_{n=1}^{\infty} \frac{1}{n!}\left(\widetilde{\frac{Q_2(s)t}{Q_1(s)}}\right)^n.$$

After elementary transformations we obtain

$$\varphi(x,t) \leq \sum_{j=1}^{l} \sum_{k=1}^{\infty} \sum_{n=[k/p_j]}^{\infty} \frac{t_k^n A_k^j(n)(2\pi)^k |x|^{k-1}}{n!(k-1)!}.$$

Taking into account that

$$|A_k^j(n)| \leq \frac{A^n 2^{(k-1)l} l(l-1) p_2 p_3 \dots p_{j-1} p_{j+1} \dots p_l \, n^{l+1}(n+1)\dots(n+k-l-1)}{(k-1)!} \tag{11}$$

for sufficiently large k, where the constants A and B are determined by the polynomials $Q_2(s)$ and $Q_1(s)$, we can write the following estimate for the remainder R_{k_0} of the series standing on the right-hand side of inequality (11):

$$R_{k_0} \leq \sum_{j=1}^{l} \sum_{k=k_0}^{\infty} \sum_{n=[k/p_j]}^{\infty} A^n B \cdot 2^{(k-1)l} l(l-1) p_1 \dots p_{j-1} p_{j+1} \dots p_l \times$$

$$\times \, n^{l+1}(n+1)(n+k-l-1)|xt|^{k-1}(n!)^{-1}[(k-1)!]^{-2}.$$

Next, using the methods for estimating positive series found by G. S. Salekhov ([7]), and the classical rule for estimating the order of growth of a power series ([7]), we obtain that the order of growth of the series standing on the right-hand side of the inequality is equal to $q/(q+1)$. Therefore

$$|\varphi(x,t)| < B_1 \exp\left[B_2 |x|^{q/(q+1)}\right].$$

Consequently, for the solution of problem (2), (2a) we obtain the formula

$$\begin{aligned} u(x,t) &= (\delta(x) + \alpha(x,t)) * (\delta(x) + \varphi(x,t)) * u_0(x) = \\ &= (\delta(x) + \alpha(x,t) + \varphi(x,t) + \alpha(x,t) * \varphi(x,t)) * u_0(x). \end{aligned} \tag{12}$$

From formula (12) for $u(x,t)$ it follows that, for the convolution to exist, it is sufficient that $u_0(x)$ satisfy inequality (3); $u(x,t)$ will then satisfy inequality (4).

The solution (12) found, obviously, satisfies equation (2) and condition (2a) as a generalized function over the corresponding space, but it is not clear whether $u(x,t)$ will also be a classical solution of this Cauchy problem, since among the nonzero generalized functions in $T(R)$ there are functionals of the type of the zero function.

To prove that $u(x,t)$ is a classical solution of problem (2), (2a), let us note that $\widehat{Q}(s,0,t)$ also belongs to the generalized functions of the space of functions $\varphi(x)$ satisfying the inequalities

$$|\varphi^{(p)}(x)| \le C_p \exp[-B|x|^{q/(q+1)}].$$

By virtue of this, $Q(s,0,t)$ defines a generalized function over the space of functions R_1 satisfying inequalities (9):

$$|\sigma^k \psi^{(p)}(\sigma)| \le C_k B^p p^{\frac{p+1}{q}p},$$

and the generalized function $Q(s,0,t)$ satisfies equation (4) over the space of functions R_1.

Applying the inverse Fourier transform, we obtain that $Q(s,0,t)$ satisfies equation (2), and, since in the space R_1 and in its Fourier transform the only zero functionals are functionals of the type of the zero function, $u(x,t)$ is also a classical solution of problem (2), (2a). The boundary found for the possible growth of solutions of equation (2), (2a) is attained.

Example. Consider the equation

$$i\frac{\partial^2 u(x,t)}{\partial t\,\partial x} = u(x,t), \qquad u(x,0) = u_0(x),$$

where $u_0(x)$ is an arbitrary finite function from R. In the present case

$$\exp\left[\frac{t}{s}\right] = \delta(x) + t + \left.\frac{2\pi J_1(2i\sqrt{2\pi xt})}{i\sqrt{2\pi xt}}\right|_{x>0} - \left.\frac{2\pi J_1(2i\sqrt{2\pi xt})}{i\sqrt{2\pi xt}}\right|_{x<0}$$

and, as is easy to show, $u(x, t)$ has order of growth $\rho = 1/2$.

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Note: Figure translations are in progress. See original paper for figures.