ON THE COMPLETENESS OF THE SYSTEM OF EIGEN AND ASSOCIATED ELEMENTS OF NON-SELF-ADJOINT OPERATORS CLOSE TO NORMAL ONES

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Abstract

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ON THE COMPLETENESS OF THE SYSTEM OF EIGEN AND ASSOCIATED ELEMENTS OF NON-SELF-ADJOINT OPERATORS CLOSE TO NORMAL ONES

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In the paper ([1]), M. V. Keldysh studied the question of the completeness of the system of eigen and associated elements of non-self-adjoint equations

$$y = \left(A_0 + \lambda H^{1/n} A_1 + \ldots + \lambda^{n-1} H^{(n-1)/n} A_{n-1} + \lambda^n H\right) y, \qquad (1)$$

where A_i $(i = 0, 1, \ldots, n-1)$ are completely continuous operators; H is a complete completely continuous self-adjoint operator of finite order. In the same paper the resolvent of equations of the form

$$y = (K_1 + \lambda K_2 + \ldots + \lambda^n K_n) y + f,$$

where K_i $(i = 1, 2, \ldots, n)$ are completely continuous operators, was investigated.

Following M. V. Keldysh, in the present note we consider the equation

$$y = \left(A + \lambda H^{1/n} A_1 + \ldots + \lambda^{n-1} H^{(n-1)/n} A_{n-1} + \lambda^n H\right) y \equiv L(\lambda) y, \qquad (2)$$

where A is a bounded operator; A_i $(i = 1, 2, \ldots, n-1)$ are completely continuous operators; H is a complete completely continuous normal operator, and it is proved that, under certain conditions imposed on A and H, the system of eigen and associated elements of equation (2) (of the operator $L(\lambda)$) is n-fold complete in the Hilbert space $\mathcal{H}$.

Everywhere we shall adhere to the terminology of the article ([1]).

Let H be a completely continuous normal operator. The **order** of the operator H is called the lower bound ρ of the numbers α for which H^α has finite absolute norm; for $\rho < \infty$, H is called an operator of finite order.

Let G be the plane of the complex variable, in which the eigenvalues of the operator λH are located.

All rays discussed below issue from the origin. We shall say that a ray belongs to the class $\mathfrak{K}_\beta$ if it is the bisector of some angle with aperture 2β, whose vertex is at the origin and inside which there may lie only a finite number of eigenvalues of the operator λH. We shall say that a ray with argument φ belongs to the class $\mathfrak{K}_\beta$ if, upon rotation through the angle $(n-1)\varphi$, it coincides with some ray of the class $\mathfrak{K}_\beta$ (the **argument of a ray** is the angle between this ray and the positive direction of the real axis). We shall say that the class of rays $\mathfrak{K}$ is ε-dense in G if inside every angle with aperture less than ε there is at least one ray from $\mathfrak{K}$.

Let A be a bounded operator; then the operator A can be represented in the form of a sum $A = A' + B'$, where A' is completely continuous and B' is a bounded operator. Such a representation is not unique. We shall call B_0 the **purely bounded part of the operator** A, if $A = A_0 + B_0$, where A_0 is a completely continuous operator, and, moreover, from the representation $A = A_1 + B_1$, where A_1 is a completely continuous operator, it follows that $\|B_1\| \geqslant \|B_0\|$. Concerning the completeness of the eigenfunctions and associated functions of equation (2), the following theorem is valid.

Theorem 1. *Let H be a completely continuous complete normal operator of finite order ρ; let A be a bounded operator with purely bounded part B_0; and let A_i $(i = 1, 2, \dots, n-1)$ be completely continuous operators.*

If, for some $\varepsilon \leqslant \pi/\rho n$, the class of rays $\mathfrak{K}^n_\beta$ for $\sin\beta > \|B_0\|$ is ε-dense in G, then the systems of eigen and associated elements of each of equations (2) and (2) (of the operators $L(\lambda)$ and $L^*(\lambda)$)*

$$y = \left[A^* + \lambda A_1^*(H^{1/n})^* + \dots + \lambda^{n-1} A_{n-1}^*(H^{(n-1)/n})^* + \lambda^n H^*\right] y \equiv L^*(\lambda) y \quad (2^*)$$

are n-fold complete in the Hilbert space $\mathfrak{H}$.

We shall precede the proof of Theorem 1 by the following lemma.

Lemma. *If the conditions of Theorem 1 are fulfilled, then the operator*

$$C(\lambda) = \left[E + T(\lambda^n)\right]\left[A + \lambda H^{1/n} A_1 + \dots + \lambda^{n-1} H^{(n-1)/n} A_{n-1}\right] \quad (3)$$

has a meromorphic resolvent $B(\lambda)$, which is bounded on all rays of $\mathfrak{K}^n_\beta$ (generally speaking, by a number depending on the ray), and the resolvents $R(\lambda)$ and $R^*(\lambda)$ of the operators $L(\lambda)$ and $L^*(\lambda)$ are representable in the form

$$R(\lambda) = T(\lambda^n) + B(\lambda)\left[E + T(\lambda^n)\right], \quad (4)$$

$$R^*(\lambda) = T^*(\lambda^n) + \left[E + T^*(\lambda^n)\right] \cdot B^*(\lambda), \quad (4')$$

where $T^*(\lambda)$ is the resolvent of the operator λH.

We outline the proof. Using the conditions of the lemma, we prove that on each ray of $\mathfrak{K}_\beta^n$, $\|C(\lambda)\| < 1 - \varepsilon$ for some $\varepsilon > 0$ ($\varepsilon < 1$) and for sufficiently large $|\lambda|$. Consequently, for sufficiently large $|\lambda|$, $B(\lambda)$ exists and is bounded on each ray of $\mathfrak{K}_\beta^n$. From (4) we obtain that $R(\lambda)$ exists (and is bounded for sufficiently large $|\lambda|$) on each ray of $\mathfrak{K}_\beta^n$. Hence it follows that $R(\lambda)$ exists on the whole plane and is a meromorphic function of λ. From (4) we have

$$B(\lambda) = [R(\lambda) - T(\lambda^n)][E - \lambda H].$$

Consequently, $B(\lambda)$ exists on the whole plane and is a meromorphic function of λ. Thus the lemma is completely proved.

For the proof of Theorem 1 we shall use the following theorem, due to M. V. Keldysh.

Theorem. *Let H be a completely continuous self-adjoint operator such that $\sum \frac{1}{|h_i|^\rho} < \infty$ (h_i are the eigenvalues of the operator λH); let $K_1, K_2, \dots, K_n$ be bounded operators. Denote by λ_i the eigenvalues of the equation*

$$y = (\lambda K_1 H + \dots + \lambda^n K_{nH}^n) y,$$

and by $R(\lambda)$ its resolvent. We have:

1. $\displaystyle\sum \frac{1}{|\lambda_j|^\rho} < \infty.$
2. $R(\lambda) = \dfrac{D(\lambda)}{\Delta(\lambda)}$, where $D(\lambda)$ is an operator function of order not exceeding ρ;

$$\Delta(\lambda) = \prod_j \left(1 - \frac{\lambda}{\lambda_j}\right) \exp\left[\sum_{i=1}^m \frac{1}{i}\left(\frac{\lambda}{\lambda_j}\right)^i\right],$$

m being the largest integer satisfying the inequality $m < \rho$.

Let us prove Theorem 1. Suppose that the theorem is false. Then there exist n elements $f_0, \dots, f_{n-1}$ such that

$$\sum_{\nu=0}^{n-1} (f_\nu, Y_h^{k,\nu}) = 0, \tag{5}$$

where $Y_h^{k,\nu}$ are the derived chains of eigen and associated elements of the operator $L(\lambda)$.

Consider the equation

$$y = [A^* + \lambda A_1^* (H^{1/n})^* + \dots + \lambda^{n-1} A^* (H^{(n-1)/n})^* + \lambda^n H^*] y + f(\lambda), \qquad (6)$$

where $f(\lambda) = f_0 + \lambda f_1 + \dots + \lambda^{n-1} f_{n-1}$.

From (6) we obtain $y = [E + R^*(\lambda)] f(\lambda)$, where $R^*(\lambda)$ is the resolvent of $L^*(\lambda)$. Using the theorem of M. V. Keldysh, it is easy to prove that $y(\lambda) = [E + R^*(\lambda)] f(\lambda)$ grows on each ray of $\mathcal{K}_\beta^n$ no faster than a polynomial of degree $n-1$. Calculating the principal parts of $R^*(\lambda) f(\lambda)$ for the function $f(\lambda) = f_0 + \lambda f_1 + \dots + \lambda^{n-1} f_{n-1}$ and using the equalities (5), we find that $[E + R^*(\lambda)] f(\lambda)$ is an entire function of λ.

By Lindelöf's theorem, taking into account the behavior of $y(\lambda)$ on the rays from $\mathcal{K}_\beta^n$, we find that $y(\lambda)$ grows in the whole plane no faster than λ^{n-1}. Hence,

$$y(\lambda) = y_0 + \lambda y_1 + \dots + \lambda^{n-1} y_{n-1}.$$

Substituting $y(\lambda)$ into equation (6) and comparing the coefficients of equal powers on the left- and right-hand sides of the equation, we conclude that all y_i $(i = 0, 1, \dots, n-1)$ are equal to zero. But this contradicts the fact that at least one of f_i $(i = 0, 1, \dots, n-1)$ is nonzero and that $y(\lambda)$ is a solution of equation (6).

Thus the theorem is proved for equation (2). The proof for (2*) is analogous.

Concerning the distribution of the eigenvalues of equation (2) (and also of (2*)), the following theorem is valid.

Theorem 2. If, under the conditions of Theorem 1, the eigenvalues of the operator λH are situated on a finite number of rays (we denote the arguments of the rays by $\varphi_1, \dots, \varphi_k$), then, for any ε for which $\sin \varepsilon > \|B_0\|$, outside the angles

$$\psi_{ij} - \varepsilon \le \arg \lambda \le \psi_{ij} + \varepsilon, \qquad (7)$$

where
$$\psi_{ij} = \frac{2\pi j + \varphi_i}{n}, \quad i = 1, 2, \dots, k; \quad j = 0, 1, \dots, n-1,$$

there can be only a finite number of eigenvalues of equation (2).

Let us outline the proof. From the definition of $\mathcal{K}_\beta^n$ it follows that outside the angles (7) all rays belong to $\mathcal{K}_\beta^n$ when $\sin \beta > \|B_0\|$. Consequently, for $\sin \varepsilon > \|B_0\|$, outside the angles (7) there can be only a finite number of eigenvalues of equation (2). If $B_0 = 0$, then we obtain that for any $\varepsilon > 0$

outside the angles (7) there can be only a finite number of eigenvalues of equation (2). Consequently, the eigenvalues of equation (2) approach asymptotically the rays

$$\arg\lambda = \psi_{ij}, \qquad i = 1, 2, \ldots, k;\ j = 0, 1, \ldots, n-1.$$

Let us note that Theorem 2 is also true in the case where the eigenvalues λH approach asymptotically the rays $\arg\lambda = \varphi_i$, $i = 1, 2, \ldots, k$.

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2. J. E. Allakhverdiev, Dokl. Acad. Sci. Azerbaijan SSR, 13, No. 5, 469 (1957).

Note: Figure translations are in progress. See original paper for figures.