Mathematics

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1957

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Abstract

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ON AN EXPRESSION FOR THE TRACE OF THE DIFFERENCE OF TWO SINGULAR DIFFERENTIAL OPERATORS OF STURM-LIOUVILLE TYPE

(Presented by Academician V. I. Smirnov on 1 III 1957)

I. M. Gel' fand and B. M. Levitan (^{1}) investigated the question of an expression for the trace of the difference of two regular operators of Sturm-Liouville type. In the case considered by them, both operators have discrete spectrum. However, in quantum field theory (^{2}) there arises the question of the finiteness of the trace of the difference of two operators, each of which has continuous spectrum. In the present work this question is considered in the simplest example of two operators of Sturm-Liouville type on the interval $(0, \infty)$. The conditions obtained and the expression for the trace coincide with those obtained in (^{1}).

1. We shall consider differential operators of the type

$$Ly = -y'' + q(x)y, \qquad y(0) = 0. \tag{1}$$

If

$$\int_0^\infty x|q(x)|\,dx < \infty,$$

then the operator L has continuous spectrum on the half-axis $0 \le \lambda < \infty$ and a finite number of negative eigenvalues $\lambda = -\varkappa_i^2$. The spectral function E_λ for $\lambda > 0$ is an integral operator with kernel $\Theta(x, y, \lambda)$, which has the derivative with respect to λ:

$$\frac{d}{d\lambda}\Theta(x, y, \lambda) = \omega(x, \lambda)\omega(y, \lambda), \tag{2}$$

where $\omega(x, \lambda)$ are eigenfunctions of the continuous spectrum normalized to $\delta(\lambda-\lambda')$. They differ only by a factor from the solutions $\varphi(x, k)$ of the equation

$$\varphi''(x,k) + k^2\varphi(x,k) = q(x)\varphi(x,k);$$
$$\varphi(0,k) = 0; \qquad \varphi'(0,k) = 1; \tag{3}$$

$$k = \sqrt{\lambda}; \qquad -\infty < k < \infty;$$

as $x \to \infty$, $\varphi(x,k)$ has the asymptotics (3):

$$\varphi(x,k) = \frac{A(k)}{k}\sin(kx - \eta(k)) + o(1). \tag{4}$$

Here $A(k)$ and $\eta(k)$ are the modulus and argument of the function $M(k)$:

$$M(k) = 1 + \int_0^\infty e^{ikx} q(x)\varphi(x,k)\,dx; \tag{5}$$

$$\eta(-k) = -\eta(k); \qquad A(-k) = A(k).$$

Lemma 1. If $q(x)$ has two continuous integrable derivatives, then, as $k \to \infty$, the following estimate holds:

$$M(k) = 1 - \frac{1}{2ik}\int_0^\infty q(t)\,dt + \frac{1}{4k^2}q(0) - \frac{1}{8k^2}\left[\int_0^\infty q(t)\,dt\right]^2 + O\left(\frac{1}{k^3}\right). \tag{6}$$

Lemma 2. If

$$\int_0^\infty x^2|q(x)|\,dx < \infty,$$

then the function $\eta(k)$ is continuously differentiable. Moreover, under this condition one may differentiate with respect to k asymptotic formulas of type (4). Thus, for the solution

$$\psi(x,k) = \frac{k}{A(k)}\varphi(x,k)$$

the following estimate is valid:

$$\dot{\psi}(x,k) = \frac{d}{dk}\psi(x,k) = (x - \dot{\eta}(k))\cos(kx - \eta(k)) + o(1). \tag{7}$$

Lemma 3. The representation

$$\ln A(k) = \frac{2}{\pi} P\int_0^\infty \frac{s\eta(s)}{s^2 - k^2}\,ds + \sum_{l=1}^{n} \ln\frac{k^2 + \varkappa_l^2}{k^2} \qquad (0 < k < \infty) \tag{8}$$

holds. Here the index P means that the integral is understood in the Cauchy sense; $-\varkappa_l^2$ are the discrete eigenvalues.

2. Let us now consider two operators with potentials $q_1(x)$ and $q_2(x)$, denoting them, respectively, by L_1 and L_2.

Define the trace of the difference $L_1 - L_2$ by the following equality:

$$\operatorname{Sp}(L_1 - L_2) \equiv \lim_{R\to\infty} \int_{-\infty}^{R} \lambda\, d[\operatorname{Sp}(E_\lambda^{(1)} - E_\lambda^{(2)})] =$$

$$= \int_{-\infty}^{0} \lambda\, d\operatorname{Sp}(E_\lambda^{(1)} - E_\lambda^{(2)}) + \lim_{R\to\infty} \int_0^R \lambda \left(\lim_{N\to\infty} \int_0^N [|\omega_1(x,\lambda)|^2 - |\omega_2(x,\lambda)|^2]\, dx \right) d\lambda.$$

It is easy to see that

$$\int_{-\infty}^{0} \lambda\, d\operatorname{Sp}(E_\lambda^{(1)} - E_\lambda^{(2)}) = -\sum_{l=1}^{n_1} \varkappa_l^{(1)2} + \sum_{l=1}^{n_2} \varkappa_l^{(2)2}.$$

Let us compute the principal part of the trace. We pass to the variable $k = \sqrt{\lambda}$. Then the second term in the expression for the trace will have the form

$$\lim_{R\to\infty} \frac{2}{\pi} \int_0^R k^2\, dk \left(\lim_{N\to\infty} \int_0^N [\psi_1^2(x,k) - \psi_2^2(x,k)]\, dx \right),$$

where $\psi(x,k) = \sqrt{\pi k}\, \omega(x,\lambda)$ is a solution of equation (3) having the asymptotic behavior

$$\psi(x,k) = \sin(kx - \eta(k)) + o(1).$$

With the aid of the differential equation for $\psi(x,k)$ and estimates of type (4) and (7), it is not difficult to obtain that

$$\int_0^N \psi^2(x,k)\, dx = \frac{1}{2}N - \frac{1}{2}\dot\eta(k) + \frac{1}{4k}\sin(2kN - \eta(k)) + o(1).$$

After integration with respect to k over finite limits and passage to the limit as $N \to \infty$, we obtain

$$\frac{2}{\pi} \int_0^R k^2 \left[\int_0^\infty (\psi_1^2(x,k) - \psi_2^2(x,k))\, dx \right] dk = -\frac{1}{\pi} \int_0^R k^2 \frac{d}{dk}(\eta_1 - \eta_2)\, dk =$$

$$= \frac{1}{\pi} R^2 (\eta_1(R) - \eta_2(R)) + \frac{2}{\pi} \int_0^R k(\eta_1(k) - \eta_2(k))\, dk.$$

Taking the logarithm of estimate (6) for $M(k)$, we obtain the estimates

$$\eta(k) = \frac{1}{2k} \int_0^\infty q(t)\, dt + O\left(\frac{1}{k^3}\right);$$

$$\ln A(k) = \frac{1}{4k^2} q(0) + O\left(\frac{1}{k^3}\right). \qquad (9)$$

If

$$\int_0^\infty [q_1(x) - q_2(x)]\, dx = 0,$$

then

$$\eta_1(k) - \eta_2(k) = O\left(\frac{1}{k^3}\right).$$

Therefore we may pass to the limit as $R \to \infty$, and as a result we obtain

$$\operatorname{Sp}(L_1 - L_2) = -\sum_{l=1}^{n_1} \chi_l^{(1)2} + \sum_{l=1}^{n_2} \chi_l^{(2)2} + \frac{2}{\pi} \int_0^\infty k(\eta_1(k) - \eta_2(k))\, dk. \qquad (10)$$

3. It is now not difficult to obtain for the trace an expression analogous to that obtained by Gelfand and Levitan. On the one hand, in view of the estimate $\eta_1(k) - \eta_2(k) = O(1/k^3)$, one can show that

$$\lim_{R\to\infty} k^2 P \int_0^\infty \frac{s(\eta_1(s) - \eta_2(s))}{s^2 - k^2}\, ds = -\int_0^\infty s(\eta_1(s) - \eta_2(s))\, ds,$$

and, consequently, taking formula (8) into account, we obtain

$$-\sum_{l=1}^{n_1} \chi_l^{(1)2} + \sum_{l=1}^{n_2} \chi_l^{(2)2} + \frac{2}{\pi} \int_0^\infty s(\eta_1(s) - \eta_2(s))\, ds = -\lim_{k\to\infty} k^2 (\ln A_1(k) - \ln A_2(k)).$$

On the other hand, from the asymptotic formula (9) for $\ln A(k)$ it follows that

$$\lim_{k\to\infty} k^2 (\ln A_1(k) - \ln A_2(k)) = \frac{1}{4}(q_1(0) - q_2(0)).$$

Finally we obtain for the trace of the difference $L_1 - L_2$ the expression

$$\operatorname{Sp}(L_1 - L_2) = -\sum_{l=1}^{n_1} \chi_l^{(1)2} + \sum_{l=1}^{n_2} \chi_l^{(2)2} +$$

$$+\frac{2}{\pi}\int_0^\infty k(\eta_1(k)-\eta_2(k))\,dk = -\frac{1}{4}(q_1(0)-q_2(0)). \tag{11}$$

This formula is the analogue of the corresponding formula of Gelfand and Levitan.

4. Formula (11) has been obtained by us under very strong restrictions on the potentials $q_i(x)$, namely, the $q_i(x)$ must be twice continuously differentiable and

$$\int_0^\infty (1+x^2)|q_i(x)|\,dx < \infty.$$

However, with the aid of a limiting passage these restrictions can be removed. More precisely, the following theorem holds:

Theorem 1. *If the following conditions are satisfied:*

a)

$$\int_0^\infty x|q_i(x)|\,dx < \infty, \qquad i = 1, 2;$$

b) $g(x) = q_1(x) - q_2(x)$ *is continuous in a neighborhood of* $x = 0$;

c)

$$\int_0^\infty g(x)\,dx = 0,$$

then the trace of the difference of the operators L_1 *and* L_2 *is finite and expression (11) holds.*

5. In the case considered above the operators had simple spectrum. The case of spectrum of finite multiplicity may be considered by the example of operators in the space of vector-valued functions with N components

$$Ly = -y'' + Q(x)y, \qquad y(0) = 0.$$

Here $Q(x)$ is a real symmetric matrix function. If

$$\int_0^\infty x\|Q(x)\|\,dx < \infty,$$

where $\|Q(x)\|$ is a suitably chosen norm, then this operator has an N-fold continuous spectrum on the half-axis $0 < \lambda < \infty$ and a finite number of negative eigenvalues of finite multiplicity. The spectral function E_λ is an integral operator with matrix kernel $\Theta(x,y,\lambda)$, which for $\lambda > 0$ has derivative

$$\frac{d\Theta(x,y,\lambda)}{d\lambda} = \frac{1}{\pi\sqrt{\lambda}}\,\Psi(x,\sqrt{\lambda})\Psi(y,\sqrt{\lambda})^*;$$

where $\Psi(x,k)$ is the matrix solution of the equation

$$\Psi(x,k)'' + k^2\Psi(x,k) = Q(x)\Psi(x,k); \qquad \Psi(0,k) = 0, \tag{12}$$

which as $x \to \infty$ has the asymptotic form

$$\Psi(x,k) = \frac{1}{2i}\{e^{ikx}I - e^{-ikx}S(k)\} + o(1). \tag{13}$$

Here I is the identity matrix; $S(k)$ is the so-called S-matrix of the operator L (this is a unitary symmetric matrix function, uniquely determined by the potential $Q(x)$ (4)).

The analogue of Theorem 1 for this case may be formulated as follows:

Theorem 2. *If the following conditions are satisfied:*

a)

$$\int_0^\infty x\|Q(x)\|\,dx < \infty, \qquad i = 1,2;$$

b) $G(x) = Q_1(x) - Q_2(x)$ *is continuous in a neighborhood of zero;*

c)

$$\int_0^\infty \operatorname{Tr} G(x)\,dx = 0,$$

where $\operatorname{Tr} G(x)$ *is the trace of the matrix* $G(x)$*,*

then the difference of the operators L_1 *and* L_2 *has a finite trace, which is expressed as follows:*

$$\operatorname{Sp}(L_1 - L_2) = -\sum_{l=1}^{n_1} m_l^{(1)}\varkappa_l^{(1)2} + \sum_{l=1}^{n_2} m_l^{(2)}\varkappa_l^{(2)2} +$$

$$+\frac{1}{\pi i}\int_0^\infty k(\ln\det S_1 - \ln\det S_2)\,dk = -\frac{1}{4}(\operatorname{Tr} Q_1(0) - \operatorname{Tr} Q_2(0)). \tag{14}$$

Here m_l is the multiplicity of the corresponding eigenvalues.

The author expresses gratitude to Prof. O. A. Ladyzhenskaya for the interest she has shown in this work.

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Received
20 II 1957

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Note: Figure translations are in progress. See original paper for figures.