# MATHEMATICS

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1957

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## Abstract

## Full Text

MATHEMATICS

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# ON THE FIRST BOUNDARY-VALUE PROBLEM FOR EQUATIONS OF ONE-DIMENSIONAL NONSTATIONARY FILTRATION

*(Presented by Academician I. G. Petrovskii on 23 II 1957)*

In the theory of one-dimensional nonstationary filtration there occur[1] equations of the form

$$\frac{\partial u}{\partial t} = \frac{\partial^2 \varphi(u)}{\partial x^2}, \tag{1}$$

where $\varphi(u) > 0,\ \varphi'(u) > 0$ for $u > 0$; $\varphi(0) = \varphi'(0) = 0$. Equation (1) for $u > 0$ is a nonlinear parabolic equation; for $u = 0$ it degenerates into an equation of the first order. In [2], for equation (1) (and also for more general equations), the Cauchy problem and the first boundary-value problem with zero boundary conditions were considered. In the present note, by the methods of [2], existence and uniqueness are proved for a generalized solution of the first boundary-value problem for equation (1) in bounded and semi-bounded domains.

Consider equation (1) in the rectangle $R\{0 \leq t \leq T,\ 0 \leq x \leq X\}$ with the initial and boundary conditions:

$$u(0, x) = f_0(x) \geq 0; \qquad u(t, 0) = f_1(t) \geq 0; \qquad u(t, X) = f_2(t) \geq 0. \tag{2}$$

Denoting by $\Gamma$ the part of the boundary of $R$ consisting of the lateral sides ($t = 0,\ x = 0$ and $x = X$), we rewrite conditions (2) in the form $u|_\Gamma = f(s)$.

**Definition 1.** A function $u(t, x) \geq 0$, continuous in $R$ and satisfying conditions (2), will be called a **generalized solution** of problem (1), (2), if there exists a generalized derivative $\partial\varphi(u)/\partial x \in L_2(R)$ and, for any function $F(t, x)$, continuously differentiable in $R$ and equal to zero for $t = T,\ x = 0,\ x = X$, the equality

$$\iint_R \left[\frac{\partial F}{\partial t} u - \frac{\partial F}{\partial x}\frac{\partial \varphi(u)}{\partial x}\right] dt\, dx + \int_0^X F(0, x) f_0(x)\, dx = 0 \tag{3}$$

holds.

**Theorem 1.** *The generalized solution of problem (1), (2) is unique.*

The proof is analogous to the proof of uniqueness in [2].

**Theorem 2.** *Let $\varphi(u)$ be 6 times continuously differentiable for $0 < u \leq M + \varepsilon = M_1$, where $M = \max f,\ \varepsilon > 0$; let the functions $\varphi(f_0(x)),\ \varphi(f_1(t)),\ \varphi(f_2(t))$ satisfy a Lipschitz condition; and let $\varphi(u)$ be subject to the requirements*

$$\varphi''(u) > 0 \quad \text{for } u > 0; \qquad \varphi'(u) = O\Big(\sqrt{\varphi''(u)}\Big) \quad \text{as } u \to 0; \qquad \int_0^u \frac{\varphi'(\alpha)}{\alpha}\,d\alpha < \infty. \tag{4}$$

*Then there exists a generalized solution $u(t,x)$ of problem (1), (2). At the points $R \setminus \Gamma$ where $u(t,x) > 0$, all derivatives of $u(t,x)$ occurring in equation (1) exist and are continuous, and $u(t,x)$ satisfies this equation.*

The assumptions of Theorem 2 are satisfied, for example, if $f_0(x),\ f_1(t),\ f_2(t)$ satisfy the Lipschitz condition and $\varphi(u) = u^\mu,\ \mu > 1$.

**Proof of Theorem 2.** Put $v = \varphi(u),\ u = \varphi^{-1}(v) = \omega(v)$. Construct on $\Gamma$ a sequence of functions $g_n(s) = \{g_{0n}(x), g_{1n}(t), g_{2n}(t)\}$ with the following properties: $g_{0n}(x),\ g_{1n}(t),\ g_{2n}(t)$ are infinitely differentiable; $g_{0n}(0) = g_{1n}(0)$; $g_{0n}(X) = g_{2n}(0)$; $g'_{1n}(0) = \varphi'(\omega(g_{0n}(0)))g''_{0n}(0)$; $g'_{2n}(0) = \varphi'(\omega(g_{0n}(X)))g''_{0n}(X)$; $0 < g_{n+1}(s) < g_n(s) \leq M_2 = \varphi(M_1)$; $\sup_{n,t,x}\{|g'_{0n}(x)|,\ |g'_{1n}(t)|,\ |g'_{2n}(t)|\} \leq N$; $g_n(s) \to \varphi(f(s))$ uniformly on $\Gamma$. As was proved in ([3]), for each $n$ there exists in $R$ a solution $v_n(t,x)$ of the equation

$$\frac{\partial v}{\partial t} = \varphi'(\omega(v))\frac{\partial^2 v}{\partial x^2} \tag{5}$$

with the condition $v_n|_\Gamma = g_n(s)$; one can show that inside $R$ all $v_n$ have continuous derivatives with respect to $x$ up to order 6. Using the maximum principle, we obtain $0 < v_{n+1} \leq v_n \leq M_2$. Hence $0 < u_{n+1} \leq u_n \leq M_1$, where $u_n = \omega(v_n)$. Consequently, there exists $\lim_{n\to\infty} u_n(t,x) = u(t,x) \geq 0$. We shall show that $u(t,x)$ is a generalized solution of the problem (1), (2).

Substitute into equation (1) its solution $u_n(t,x)$. Multiplying both sides by the function

$$F_n(t,x) = \varphi(u_n(t,x)) - g_{1n}(t) + \frac{x}{X}\,[g_{1n}(t) - g_{2n}(t)]$$

and integrating by parts, we obtain for the integral

$$\iint_R \left[\frac{\partial \varphi(u_n)}{\partial x}\right]^2 dt\,dx$$

an estimate independent of $n$; hence the sequence $\{\partial\varphi(u_n)/\partial x\}$ is bounded in $L_2(R)$. It follows that there exists a generalized derivative $\partial\varphi(u)/\partial x \in L_2(R)$, and a subsequence $\{u_{n_k}\}$ can be found such that $\{\partial\varphi(u_{n_k})/\partial x\}$ converges weakly in $L_2(R)$ to $\partial\varphi(u)/\partial x$ (see ([4]), pp. 42-43).

Multiply both sides of (1) by a function $F(t,x)$, continuously differentiable in $R$ and equal to zero for $t = T$, $x = 0$, $x = X$; integrating by parts, we obtain that for $u_{n_k}$ equality (3) is satisfied. Passing to the limit, we establish that (3) is valid also for $u(t,x)$.

We shall show that in every rectangle $R_\delta\{0 \le t \le T,\ 0 \le \delta \le x \le X - \delta\}$ the derivatives $\partial\varphi(u_n)/\partial x = \partial v_n/\partial x$ are uniformly bounded in modulus. For this we use the method of S. N. Bernstein, similarly to how this was done in the work ([5]). Consider in $R$ the function

$$w_n(t,x) = x(X-x)|\partial v_n/\partial x| + e^{2Xv_n}.$$

If the maximum of $w_n$ is attained when $\partial v_n/\partial x = 0$ or on $\Gamma$, then $w_n \le NX^2/4 + e^{2XM_2} = K_1$. If, however, the maximum of $w_n$ is attained in $R\setminus\Gamma$, and at the point of maximum $w_n$, $\partial v_n/\partial x \ne 0$, then at this point

$$\varphi'(\omega(v_n))\partial^2 w_n/\partial x^2 - \partial w_n/\partial t \le 0;$$

using equation (5) and the equation obtained by differentiating (5) with respect to $x$, and also taking into account that $\partial w_n/\partial x = 0$ at the point of maximum of $w_n$, we bring the last inequality to the form

$$\left\{4X^2 e^{2Xv_n}\varphi' + \frac{\varphi''}{\varphi'}\left[2Xe^{2Xv_n} \pm (X-2x)\right]\right\}\left(\frac{\partial v_n}{\partial x}\right)^2 +$$

$$+\, 2\varphi'\left\{\mp 1 - \frac{(X-2x)[2Xe^{2Xv_n} + (X-2x)]}{x(X-x)}\right\}\frac{\partial v_n}{\partial x} \le 0.$$

By virtue of conditions (4) we obtain from this $|\partial v_n/\partial x| \le K_2/x(X-x)$, $w_n \le K_2 + e^{2XM_2} = K_3$ at the point of maximum of $w_n$; all the more so at every other point $w_n \le K_3$. Taking $K = \max(K_1, K_3)$, we find that in $R_\delta$

$$|\partial v_n/\partial x| \le K/\delta(X-\delta) = K_\delta.$$

The function $\varphi(u(t,x))$ satisfies in $R_\delta$ a Lipschitz condition with respect to $x$ with constant $K_\delta$, independent of $t$. As in ([2]), we shall prove that $\varphi(u(t,x))$

is continuous in any $R_\delta$. Consequently, $u(t,x)$ is continuous everywhere for $0 \le t \le T,\ 0 < x < X$. In particular,

$$\lim_{(t,x)\to(0,x_0)} u(t,x) = f_0(x_0)$$

for $0 < x_0 < X$.

Let us prove the continuity of $u(t,x)$ at the point $(t_0,0)$, $0<t_0\leq T$. From the fact that $v_n \geq v_{n+1}$ and $v_n(t_0,0)\to\varphi(f_1(t_0))$, it follows that

$$\overline{\lim}_{(t,x)\to(t_0,0)} u(t,x)\leq f_1(t_0). \tag{6}$$

Since $u(t,x)\geq 0$, in the case $f_1(t_0)=0$ we obtain from (6)

$$\lim_{(t,x)\to(t_0,0)} u(t,x)=0=f_1(t_0).$$

Let now $f_1(t_0)>0$. As G. I. Barenblatt showed ([6]), equation (1) for $t\geq t_1,\ x\geq 0$ has solutions of the form

$$\tilde{u}(t,x)=\begin{cases}\Phi^{-1}(c[c(t-t_1)-x]), & 0\leq x\leq c(t-t_1),\\ 0, & x\geq c(t-t_1).\end{cases}$$

Here

$$\Phi(u)=\int_0^u \frac{\varphi'(a)}{a}\,da;$$

by virtue of (4) the integral exists.

It is not hard to verify that $\tilde{u}(t,x)$ in any rectangle will be a generalized solution of equation (1) in the sense of Definition 1. Take arbitrary $\eta>0,\ \delta>0$. By choosing $c$ and $t_1$ in a suitable way, we obtain $\tilde{u}(t_0,0)=f_1(t_0)-\eta$, and, for sufficiently small $\gamma>0$, on the boundary $\tilde{\Gamma}(t=t_1,\ x=0,\ x=\delta)$ of the rectangle $\tilde{R}\{t_1\leq t\leq\min(t_0+\gamma,T),\ 0\leq x\leq\delta\}$ we have

$$u_n|_{\tilde{\Gamma}}>\tilde{u}|_{\tilde{\Gamma}} \qquad (n=1,2,...).$$

Construct on $\tilde{\Gamma}$ a monotonically decreasing sequence of smooth functions $\tilde{f}_n$, uniformly convergent to $\tilde{u}|_{\tilde{\Gamma}}=\tilde{f}$; let $\{\tilde{u}_n\}$ be the corresponding sequence of solutions of equation (1) with the condition $\tilde{u}_n|_{\tilde{\Gamma}}=\tilde{f}_n$. Subtracting one of the other equalities of the form (3), written for $\tilde{u}_n$ and $\tilde{u}$, we find

$$\iint_{\tilde{R}}\left\{\frac{\partial F}{\partial t}(\tilde{u}_n-\tilde{u})-\frac{\partial F}{\partial x}\left[\frac{\partial\varphi(\tilde{u}_n)}{\partial x}-\frac{\partial\varphi(\tilde{u})}{\partial x}\right]\right\}dt\,dx=$$

$$=\int_0^\delta F(t_1,x)\,[\tilde{u}(t_1,x)-\tilde{u}_n(t_1,x)]\,dx.$$

Put in this equality

$$F(t,x) = \int_{t_0+\gamma}^{t} \Big\{ [\varphi(\tilde u_n(\tau,x)) - \varphi(\tilde u(\tau,x))] + \Big(\frac{x}{\delta} - 1\Big) [\varphi(\tilde u_n(\tau,0)) - \varphi(\tilde u(\tau,0))] - \frac{x}{\delta}[\varphi(\tilde u_n(\tau,\delta)) - \varphi(\tilde u(\tau,\delta))] \Big\}$$

After transformations we obtain

$$\lim_{n\to\infty} \iint_{\tilde R} [\varphi(\tilde u_n) - \varphi(\tilde u)](\tilde u_n - \tilde u)\, dt\, dx = 0.$$

Hence it follows that $\tilde u_n(t,x) \to \tilde u(t,x)$ at every point of $\tilde R$.

Since $u_n|_{\tilde\Gamma} > \tilde u|_{\tilde\Gamma}$, we may assume that $u_n|_{\tilde\Gamma} > \tilde u_n|_{\tilde\Gamma}$; by the maximum principle $u_n(t,x) \geq \tilde u_n(t,x)$ everywhere in $\tilde R$. Thus, $u(t,x) \geq \tilde u(t,x)$, whence

$$\underline{\lim}_{(t,x)\to(t_0,0)} u(t,x) \geq \tilde u(t_0,0) = f_1(t_0) - \eta. \tag{7}$$

In view of the arbitrariness of $\eta$, (6) and (7) give

$$\lim_{(t,x)\to(t_0,0)} u(t,x) = f_1(t_0).$$

The proof of the continuity of $u(t,x)$ at the point $(t_0, X)$, $0 < t_0 \leqslant T$, reduces to the case already considered by the substitution $x' = X - x$. The continuity of $u(t,x)$ at the points $(0,0)$ and $(0,X)$ can be proved with the aid of ordinary barriers (see (7), pp. 344-345). Finally, the existence, at any point of $R \setminus \Gamma$ where $u(t,x) > 0$, of continuous derivatives of $u(t,x)$ entering into equation (1) is obtained as a consequence of the uniform boundedness, in a neighborhood of such a point, of the successive derivatives of $v_n(t,x)$ with respect to $x$ up to the 4th order, which is established by the method of S. N. Bernstein. Theorem 2 is proved.

**Corollary.** The generalized solution $u(t,x)$ of problem (1), (2) will be positive in $R \setminus \Gamma$ and therefore will have everywhere in $R \setminus \Gamma$ continuous derivatives entering into equation (1), if (in addition to the assumptions of Theorem 2) $f_0(x) > 0$ for $0 < x < X$.

Indeed, consider in $R$ the function

$$z(t,x) = \alpha_\delta e^{-M_3\pi^2 t/(X-2\delta)^2} \sin\frac{\pi(x-\delta)}{X-2\delta},$$

where $M_3 = \varphi'(M_1)$, $\alpha_\delta = \inf_{\delta<x<X-\delta} f_0(x)$. Obviously,

$$\partial z/\partial t = M_3 \partial^2 z/\partial x^2$$

everywhere in $R$; $z < 0$ in $R \setminus R_\delta$; $z > 0$, $\partial^2 z/\partial x^2 < 0$ inside $R_\delta$. Taking these properties of $z$ into account, it is easy to prove, by means of the maximum principle, that everywhere in $R$

$$v_n(t,x) \geqslant z(t,x), \qquad n = 1, 2, \ldots$$

Consequently, in $R_{2\delta}$, $u(t,x) \geqslant \beta_\delta > 0$.

Let us now consider equation (1) in the half-strip $D\{0 \leqslant t \leqslant T,\ 0 \leqslant x < \infty\}$ under the boundary conditions

$$u(0,x) = f_0(x) \geqslant 0, \qquad u(t,0) = f_1(t) \geqslant 0. \tag{8}$$

**Definition 2.** A function $u(t,x) \geqslant 0$, continuous and bounded in $D$, satisfying conditions (8), will be called a **generalized solution of problem (1), (8)** if there exists a generalized derivative $\partial\varphi(u)/\partial x$, square-summable in any finite domain and bounded for $0 \leqslant t \leqslant T$, $\delta \leqslant x < \infty$ (whatever $\delta > 0$ may be), and if, for every continuously differentiable function $F(t,x)$, equal to zero for $x = 0$, $t = T$, and different from zero only in a finite domain, the equality

$$\iint_D \left[\frac{\partial F}{\partial t} u - \frac{\partial F}{\partial x}\frac{\partial \varphi(u)}{\partial x}\right] dt\,dx + \int_0^\infty F(0,x) f_0(x)\,dx = 0$$

holds.

**Theorem 3.** *The generalized solution of problem (1), (8) is unique.*

The proof is carried out in the same way as in ([2]).

**Theorem 4.** *Let $f_0(x) \leqslant M$, $f_1(t) \leqslant M$, and let the functions $\varphi(f_0(x))$ and $\varphi(f_1(t))$ satisfy a Lipschitz condition respectively for $0 \leqslant x < \infty$ and $0 \leqslant t \leqslant T$, while for $\varphi(u)$ the conditions indicated in Theorem 2 are fulfilled. Then there exists a generalized solution of problem (1), (8). At those points of $D$ where $t > 0$, $x > 0$, and $u(t,x) > 0$, all derivatives of $u(t,x)$ entering into equation (1) exist and are continuous, and $u(t,x)$ satisfies this equation.*

The proof is similar to the proof of Theorem 2.

The author thanks O. A. Oleinik for guidance and assistance.

Mathematical Institute named after V. A. Steklov
Academy of Sciences of the USSR

Received
14 II 1957

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*Note: Figure translations are in progress. See original paper for figures.*