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Abstract

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ON THE GENERAL THEORY OF IRREGULAR WAVEGUIDES

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1. In ([1]) a method was proposed that reduces the problem of the propagation of radio waves in an irregular waveguide to a system of ordinary differential equations. For waveguides with slowly varying parameters this method makes it possible, in a number of cases, to give an explicit solution. In ([2–4]) its effectiveness was shown for the calculation of specific waveguide transitions. In ([1]) the method was applied to a problem in which the potential function in an irregular waveguide satisfied, in each transverse section, the same boundary condition as the potential function in a regular waveguide with the same section. Below it is applied to the problem of a symmetric electric wave in a symmetric waveguide transition of circular cross section, in which the indicated restriction is removed.

Maxwell' s equations in cylindrical coordinates, with time dependence $\exp(i\omega t)$, give for the field under consideration (a prime denotes $\partial/\partial z$)

$$E_r = -\frac{1}{ik}H', \quad E_z = \frac{1}{ik}\frac{1}{r}\frac{\partial}{\partial r}(rH); \quad k = \frac{\omega}{c}, \tag{1}$$

where $H(r, z)$ is the azimuthal component of the magnetic field, satisfying the equation

$$\widetilde{\nabla}^2 H + H'' + k^2 H = 0, \quad \widetilde{\nabla}^2 H \equiv \frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}(rH)\right]. \tag{2}$$

The transverse sections of the waveguide by the planes $z = \text{const}$ are circles of radius $a(z)$. On the metal (assumed perfectly conducting), at $r = a$, H must satisfy the boundary condition

$$\frac{1}{r}\frac{\partial}{\partial r}(rH) - a'H' = 0.$$

Introduce functions $H_m(r, z)$, representing, for any $z = \text{const}$, the magnetic field of waves E_{0m} in a regular waveguide with radius $a(z)$. These functions satisfy the equations

$$\widetilde{\nabla}^2 H_m + \delta_m^2(z) H_m = 0 \tag{3}$$

and the boundary condition $\partial(rH_m)/\partial r = 0$. They are equal to $NJ_1(\delta_m r)$, where $J_0(\delta_m a) = 0$, and N is determined from the normalization condition. Taking into account the orthogonality of H_m, we write this condition in the form $\int H_m H_p\, dS = \delta_{mp}$, where the integral is taken over the transverse section.

Represent, analogously to (1), H in the form of the series

$$H = \sum_m (A_m e^{-i\lambda_m} + C_m e^{i\lambda_m})\, r_m^{-1/2} H_m, \tag{4}$$

where

$$r_m(z) = (k^2 - \delta_m^2)^{1/2}, \quad \lambda_m(z) = \int_0^z r_m\, dz. \tag{5}$$

We shall relate the sought functions $A_m(z)$ and $C_m(z)$ and their derivatives by an additional condition, obtained when the equality

$$H' = i \sum (-A_m e^{-i\lambda_m} + C_m e^{i\lambda_m})\, r_m^{1/2} H_m. \tag{6}$$

is satisfied.

When substituting (4) into (2), one cannot apply the operator $\widetilde{\nabla}^2$ to each term of the series separately—this is the principal difference between the problem under consideration and ([1]). To determine $\widetilde{\nabla}^2 H$ we shall use a known device ([5,6]), expanding this quantity in a series in H_m and relating to one another the coefficients of the resulting expansion and of expansion (4) by means of Green' s formula for the operator $\widetilde{\nabla}^2$, applied to the functions H and H_m. Taking into account (3) and the boundary conditions for H and H_m, we obtain

$$\widetilde{\nabla}^2 H = \sum \left[-\delta_m^2\, (A_m e^{-i\lambda_m} + C_m e^{i\lambda_m})\, r_m^{-1/2} + a' \oint H' H_m\, ds \right] H_m, \tag{7}$$

where the last integral is taken over the contour of the cross section.

Substituting (6) and (7) into (2), we obtain a second condition relating A_m, C_m and their derivatives; solving this condition and the condition following from (6) with respect to A'_m and C'_m, we find the required system of equations, analogous to system (12) of ([1]):

$$A'_p = C_p e^{2i\lambda_p} \frac{r'_p}{2r_p} - \frac{ia' e^{i\lambda_p}}{2\sqrt{r_p}} \oint H' H_p\, ds - \frac{e^{i\lambda_p}}{2\sqrt{r_p}} \sum_m \gamma_{mp} \left[(r_p + r_m) A_m e^{-i\lambda_m} + (r_p - r_m) C_m e^{i\lambda_m} \right] r_m^{-1/2}; \tag{8}$$

$$C_p' = A_p e^{-2i\lambda_p} \frac{r_p'}{2r_p} + \frac{ia' e^{-i\lambda_p}}{2\sqrt{r_p}} \oint H' H_p \, ds - \frac{e^{-i\lambda_p}}{2\sqrt{r_p}} \sum_m \gamma_{mp} \left[(r_p - r_m) A_m e^{-i\lambda_m} + (r_p + r_m) C_m e^{i\lambda_m} \right] r_m^{-1/2}. \tag{8a}$$

Here $\gamma_{mp} = \int H_m' H_p \, dS$. For these quantities, from (4) and the normalization condition it is easy to obtain

$$\gamma_{mm} = -\frac{a'}{2} \oint H_m^2 \, ds = -\frac{a'}{a};$$

$$\gamma_{mp} = -\frac{\delta_m^2}{\delta_p^2 - \delta_m^2} a' \oint H_m H_p \, ds = \frac{\delta_m^2}{\delta_p^2 - \delta_m^2} \frac{2a'}{a}. \tag{9}$$

The system (8) must be solved under definite conditions at the ends $z = 0$ and $z = L$ of the irregular section. If from the left there is incident a wave to which we assign the number $m = 1$, then these conditions are as follows:

$$A_1 = 1, \qquad A_m = 0, \quad m \neq 1 \text{ at } z = 0; \qquad C_m = 0 \text{ at } z = L. \tag{10}$$

For a smooth transition ($a' \ll 1$), to which we shall restrict ourselves here, the field in any section is close to the field of the incident wave, and when evaluating in (8) the contour integrals, which are preceded by the small factor a', one may replace the total field by the field of this incident wave. Repeating further the corresponding considerations of ([1]), we obtain from (8), (10), for the sought amplitudes of the outgoing parasitic waves, with accuracy up to terms of second order in a',

$$\mp \int_0^L \frac{1}{\sqrt{r_1 r_m}} e^{-i(\lambda_1 \mp \lambda_m)} \frac{k^2 \mp r_1 r_m}{r_m \mp r_1} \frac{a'}{a} \, dz. \tag{11}$$

With the upper sign, expression (11) gives $A_m(L)$ ($m \neq 1$); with the lower sign it gives $C_m(0)$ ($m = 1, 2, ...$). The amplitude A_1 remains constant. According to (1), (6), and the normalization condition for H_m, this means that, as should follow from the law of conservation of energy, the energy of the forward wave $m = 1$ is not changed, to this order, along the irregular waveguide.

In the method developed above and in ([1]), problems of matching different waveguides are solved more simply than in other known methods ([6,7]); the method directly gives the amplitudes of the outgoing parasitic waves. It can be generalized to any smooth waveguide transition. In the general case the fields are expressed in terms of two potential functions, for which expansions of the type (4), (6), and (7)* are used.

2. Here we shall indicate another method** for obtaining formulas of the type (11), proposed in ([2]) and applied to a bent waveguide in ([8]). It is not fully rigorous, but is very visual and makes it possible, by elementary means, to obtain all the most interesting characteristics of an inhomogeneous waveguide.

The parasitic wave formed in an irregular waveguide is the sum of elementary parasitic waves formed on small intervals dz, into which the whole waveguide may be divided, as the incident wave passes through them. The propagation of the incident wave along the inhomogeneous waveguide from $z = 0$ to the place where the small interval under consideration is located, and the propagation of the elementary parasitic wave from the place of its formation to $z = L$ (for forward parasitic waves) and to $z = 0$ (for backward parasitic waves), proceed according to the laws of geometrical optics. The change of phase is determined by the factor $\exp(\pm i\lambda_m)$ for electric waves and $\exp(\pm i\gamma_m)$ for magnetic waves, where γ_m is related to the wave number $h_m = (k^2 - \alpha_m^2)^{1/2}$ of the magnetic wave in a regular waveguide in the same way as λ_m is related to r_m (5). The amplitude of the wave changes in such a way that the energy flux of the wave is conserved; if the fields are expressed through the potential functions φ_m (electric waves) and ψ_m (magnetic waves), normalized by the condition $\int(\nabla\varphi_m)^2\,dS = 1$, $\int(\nabla\psi_m)^2\,dS = 1$, then the coefficients multiplying these functions are proportional to $r_m^{-1/2}$ and $h_m^{-1/2}$ (cf. (1) and (4)). The formation of a parasitic wave on a small irregular section of length dz takes place in the same way ([9]) as at the junction of two regular waveguides, if the height $l(s)$ of the step between these waveguides is equal to $\vartheta(s)\,dz$, where $\vartheta(s)$ is the angle made by the tangent to the waveguide, perpendicular to the cross section, with the z-axis. The problem of matching two waveguides has been solved for an arbitrary function $l(s)$ in ([9])***. When, for example, an electric wave $m = 1$ is incident, the amplitudes of the electric waves that arise, according to ((9), (35)), will be

$$\pm\frac{k^2 \mp r_1 r_m}{2\sqrt{r_1 r_m}\,(r_m \mp r_1)} \oint \frac{\partial\varphi_1}{\partial n}\,\frac{\partial\varphi_m}{\partial n}\,l(s)\,ds, \tag{12}$$

where the upper sign corresponds to the forward wave and the lower sign to the backward wave. In contrast to ([9]), the same normalization of amplitudes is adopted here as throughout the present paper and in ([1]); with this normalization the ratio of the energies carried by two waves is equal to the ratio of the squares of the moduli of their amplitudes.

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* One may include in the calculation waveguides with inhomogeneous dielectric filling and solve problems of mutual compensation of inhomogeneities of various kinds, as well as problems with other cases of excitation of inhomogeneous waveguides.

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*** As we have shown elsewhere, this problem is solved much more simply by using the equivalent boundary condition proposed in ([10]), i.e., by replacing the deformation with equivalent magnetic currents.

According to the preceding, the amplitudes of the parasitic electric waves arising at the waveguide transition will be equal to

$$\pm \int_0^L \frac{1}{2\sqrt{r_1 r_m}} e^{-i(\lambda_1 \mp \lambda_m)} \frac{k^2 \mp r_1 r_m}{r_m \mp r_1} \oint \vartheta \frac{\partial \varphi_1}{\partial n} \frac{\partial \varphi_m}{\partial n} \, ds \, dz. \tag{13}$$

The amplitudes of the magnetic waves will be

$$\int_0^L \frac{k}{2\sqrt{r_1 h_m}} e^{-i(\lambda_1 \mp \gamma_m)} \oint \vartheta \frac{\partial \varphi_1}{\partial n} \frac{\partial \psi_m}{\partial s} \, ds \, dz. \tag{14}$$

When a magnetic wave is incident, according to ((9), (8)), the amplitudes of the electric waves will be

$$-\int_0^L \frac{k}{2\sqrt{h_1 r_m}} e^{-i(\gamma_1 \mp \lambda_m)} \oint \vartheta \frac{\partial \varphi_m}{\partial n} \frac{\partial \psi_1}{\partial s} \, ds \, dz \tag{15}$$

and the amplitudes of the magnetic waves

$$\pm \int_0^L \frac{1}{2\sqrt{h_1 h_m}} e^{-i(\gamma_1 \mp \gamma_m)} \frac{1}{h_m \mp h_1} \left\{ \alpha_1^2 \alpha_m^2 \oint \vartheta \psi_1 \psi_m \, ds - (k^2 \mp h_1 h_m) \oint \vartheta \frac{\partial \psi_1}{\partial s} \frac{\partial \psi_m}{\partial s} \, ds \right\} dz. \tag{16}$$

Particular cases of these general formulas are (11) and (1), (18).

Expressions (13)–(16) often (cf. ([4,3])) can be evaluated by integration by parts. For example, the reflection coefficients from a cone ($a' = \text{const}$) connecting two circular-cross-section waveguides will be, for electric (E_{nm}) and magnetic (H_{nm}) waves, respectively:

$$\frac{-ia'}{4} \left| \frac{k^2 + r^2}{r^3 a} e^{-2i\lambda} \right|_0^L, \qquad \frac{-ia'}{4(\mu^2 - n^2)} \left| \frac{\mu^4 - n^2 a^2 (k^2 + h^2)}{h^3 a^3} e^{-2i\gamma} \right|_0^L \tag{17}$$

(cf.: ([9]), (38), (31)), where $\lambda(L)$ and $\gamma(L)$ are easily calculated (cf. ([3])), and μ is the m-th root of the equation $J_n'(\mu) = 0$.

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Note: Figure translations are in progress. See original paper for figures.