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Abstract

Full Text

MATHEMATICS

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COMPLETENESS OF THE SYSTEM OF EIGENFUNCTIONS AND ASSOCIATED FUNCTIONS OF STRONGLY ELLIPTIC SYSTEMS OF DIFFERENTIAL EQUATIONS

(Presented by Academician S. L. Sobolev on 12 XI 1956)

Let x be a point of a bounded domain D of n -dimensional Euclidean space, bounded by a smooth boundary Γ ; let $u(x) = (u_1(x), u_2(x), \dots, u_N(x))$ be a vector-valued function. The system of equations

$$\begin{aligned}
 Lu = & (-1)^m \sum_{(i),(j)} \frac{\partial^m}{\partial x_{i_1} \dots \partial x_{i_m}} \left(B^{(i_1 \dots i_m; j_1 \dots j_m)}(x) \frac{\partial^m u}{\partial x_{j_1} \dots \partial x_{j_m}} \right) \\
 & + \mu \sum_{(i),(j)} \frac{\partial^m}{\partial x_{i_1} \dots \partial x_{i_m}} \left(K^{(i_1 \dots i_m; j_1 \dots j_m)}(x) \frac{\partial^m u}{\partial x_{j_1} \dots \partial x_{j_m}} \right) + T(x)u = f, \\
 & i_1, \dots, i_m; j_1, \dots, j_m = 1, 2, \dots, n
 \end{aligned} \tag{1}$$

(where $B^{(i_1 \dots i_m; j_1 \dots j_m)}(x)$, $K^{(j_1 \dots j_m; i_1 \dots i_m)}$ are complex matrices of order N , continuously differentiable m times; $T(x)$ is a differential expression of order less than $2m$) is called strongly elliptic if:

1. For any ξ_1, \dots, ξ_n such that $\sum_{i=1}^n \xi_i^2 = 1$, the matrix

$$\sum_{(i),(j)} B^{(i_1 \dots i_m; j_1 \dots j_m)}(x) \xi_{i_1} \dots \xi_{i_m} \xi_{j_1} \dots \xi_{j_m}, \quad i_1, \dots, i_m; j_1, \dots, j_m = 1, 2, \dots, n$$

is positive definite for every $x \in D$.

2. The matrices $B^{(i_1 \dots i_m; j_1 \dots j_m)}(x)$ are Hermitian for every $x \in D$.
3. The matrices $K^{(i_1 \dots i_m; j_1 \dots j_m)}(x)$ are skew-symmetric for every $x \in D$.
4. $B^{(i_1 \dots i_m; j_1 \dots j_m)}(x) = B^{(i_1 \dots i_m; i_2 \dots i_m)}(x)$ for any set $i_1 \dots i_m; j_1 \dots j_m$.

Any system of the form

$$Lu = (-1)^m \sum_{(i)} A^{(i_1 i_2 \dots i_{2m})}(x) \frac{\partial^{2m} u}{\partial x_{i_1} \dots \partial x_{i_{2m}}} + T_1(x)u = f,$$

for which the matrix

$$\sum_{(i)} (A^{(i_1 \dots i_{2m})}(x) + A^{(i_1 \dots i_{2m})^*}(x)) \xi_{i_1} \dots \xi_{i_{2m}}$$

is positive definite (1), can be reduced to system (1).

The boundary conditions

$$\left. \frac{\partial^r u}{\partial x_{i_1} \dots \partial x_{i_r}} \right|_{S_{n-k}} = 0, \quad r = 0, 1, \dots, m - \left[\frac{k}{2} \right] - 1, \quad (2)$$

where S_{n-k} is a manifold of dimension $n - k$ bounding the domain D , are called zero boundary conditions of Dirichlet type. In particular, when the boundary of the domain D is nondegenerate, they mean that $u(x)$ vanishes on the boundary together with all derivatives up to order $m - 1$.

Definition. A chain of functions $u_0, u_1, \dots, u_k, \dots$ is called a chain of eigenfunctions and associated functions of the Dirichlet problem for equations (1), (2), corresponding to the eigenvalue λ , if $u_0 \neq 0$ and the relations

$$\begin{aligned} Lu_0 - \lambda u_0 &= 0, \\ Lu_1 - \lambda u_1 &= u_0, \\ &\dots \\ Lu_k - \lambda u_k &= u_{k-1}, \\ &\dots \end{aligned} \quad (3)$$

hold, and if all the functions $u_0, u_1, \dots, u_k, \dots$ satisfy the boundary conditions (2).

Definition. The system of all eigenfunctions and associated functions of problem (1), (2) is called complete in the space $L^2(D)$ of vector-functions $u(x)$ with square summable over the domain D , if the linear span of this system is everywhere dense in $L^2(D)$.

Lemma 1. *The operator H , generated by problem (1), (2) and acting in $L^2(D)$, and the operator H^2 have the same linear spans of all eigenfunctions and associated functions.*

Lemma 2. *If B is a completely continuous operator of Hilbert-Schmidt type, then its resolvent is representable as the ratio of two entire functions, each of which has order of growth not exceeding the second ⁽³⁾:*

$$(E - \lambda B)^{-1} = \frac{D(\lambda)}{\Delta(\lambda)}, \quad \|D(\lambda)\| \leq e^{M|\lambda|^2}, \quad |\Delta(\lambda)| \leq e^{M_1|\lambda|^2}.$$

Lemma 3. *If B is a completely continuous operator of Hilbert-Schmidt type, then the resolvent of the operator B^2 is representable as the ratio of an entire operator-valued function $D(\lambda)$ to an entire function $\Delta(\lambda)$, each of which has order of growth not exceeding the first.*

Raise the system (1), (2) successively p times to a square. Then, introducing operators of gradient and divergence type, one can write the system of equations obtained from (1), (2) in the form

$$[(G^*G)^n + T_n(x)]u = f,$$

where $T_n(x)$ is an operator of order $\leq 2m \cdot 2^p$, $n = 2^p$ (1).

Now the problem for the eigenfunctions and associated functions of the operator L^n can be written in the following form:

$$(E + \mu A - \lambda B_p)g_n = B_{pg_{n-1}}, \quad (4)$$

where $A = (G^{-1}G^{*-1} \dots G^{-1}G^{*-1})T_n(x)(G^{-1}G^{*-1} \dots G^{-1}G^{*-1})$ is a bounded operator; $B_p = (G^{-1}G^{*-1})^n$, $g_n = (G^*G)^{n/2}u_n \in \mathfrak{H}$ (\mathfrak{H} is the space of functions $(G^*G)^{n/2}u$).

Lemma 4. *For sufficiently large p , the operator B_p is an operator of Hilbert-Schmidt type.*

Idea of the proof. Compare the system (1), (2) with the system $\Delta^m u = f$ with the corresponding boundary conditions. The latter system generates the operators G_Δ, G_Δ^* , and the metric in the space of gradients is equivalent to the metric in the space of gradients $G u$. For large n , the operator $(G_\Delta^{-1}G_\Delta^{*-1})^n$ is an operator of Hilbert-Schmidt type by virtue of the properties of the Green function of the Dirichlet problem. The lemma follows from this.

Find p from Lemma 4. The operator B_p satisfies the conditions of Lemma 2, and the operator B_{p+1} the conditions of Lemma 3. Hence the operator B_{p+1} has the resolvent $D(\lambda)/\Delta(\lambda)$, where $\|D(\lambda)\| < e^{M|\lambda|}$, $|\Delta(\lambda)| < e^{M_1|\lambda|}$.

Lemma 5. *The spectrum of equation (4) is discrete.*

Theorem. *Let the operator B have the following properties:*

1. From $(Bu, u) = 0$ it follows that $u = 0$.
2. The operator B is self-adjoint, completely continuous, with a resolvent whose numerator and denominator may be taken to be of growth order not higher than one.

Suppose, moreover, that the operator A is bounded and $(E + \mu A)^{-1}$ is bounded. Then the system of eigenvectors and associated vectors of the equation

$$(E + \mu A - \lambda B)f_n = Bf_{n-1}, \quad n = 0, 1, \dots,$$

is complete in the space \mathfrak{H} for sufficiently small μ .

Proof. Since the operators $E + \mu A$ and $(E + \mu A)^{-1}$ are bounded, they map every dense set into a dense set. Making the substitution $(E + \mu A)f = g$, it is enough to show that the system of eigenvectors and associated vectors of the equation

$$(E - \lambda B(E + \mu A)^{-1})g_n = B(E + \mu A)^{-1}g_{n-1}, \quad n = 0, 1, \dots,$$

is complete in \mathfrak{H} , or, denoting $B(E + \mu A)^{-1} = T$, of the equation

$$(E - \lambda T)g_n = Tg_{n-1}.$$

Assume that the system of eigenvectors and associated vectors of the last equation is not complete. This means that there is a subspace $\Omega \subseteq \mathfrak{H}$, orthogonal to all eigenvectors and associated vectors of equation (5). It is invariant with respect to T^* , and $E - \lambda T^*$ has a bounded inverse in Ω . Consider the part T_Ω^* of the operator T^* lying in Ω . Since the numerator and denominator of the resolvent of the operator B are entire functions of growth order not higher than one, the resolvents of the operators T, T^*, T_Ω^* possess the same property. The operator T_Ω^* has no eigenfunctions, and therefore its resolvent has no poles and is an entire function of growth order not higher than one:

$$\|(E - \lambda T_\Omega^*)^{-1}\| \leq e^{M|\lambda|}.$$

We shall now estimate the growth of the resolvent of the operator T_Ω^* more precisely, using the fact that the operator B is self-adjoint. For this purpose write:

$$T^* = (E + \mu A^*)^{-1}B, \quad B = (E + \mu A^*)T^*,$$

$$E - \lambda B = E - \lambda(E + \mu A^*)T^* + \mu A^* - \mu A^* = (E + \mu A^*)(E - \lambda T^*) - \mu A^*.$$

Assuming that λ is not an eigenvalue of the operators B and T^* , multiply the preceding equality on the left by $(E - \lambda B)^{-1}$ and on the right by $(E - \lambda T^*)^{-1}$. We obtain:

$$(E - \lambda T^*)^{-1} = (E - \lambda B)^{-1}(E + A^*) - \mu(E - \lambda B)^{-1}A^*(E - \lambda T^*)^{-1};$$

taking the norm of both sides of the equality, we obtain the inequality:

$$\|(E - \lambda T_\Omega^*)^{-1}\| \leq \frac{|\lambda|}{|\operatorname{Im} \lambda|} \|E + A^*\| + \frac{|\lambda|}{|\operatorname{Im} \lambda|} \|\mu A^*\| \cdot \|(E - \lambda T_\Omega^*)^{-1}\|.$$

Denote by α the angle in which

$$\frac{|\lambda|}{|\operatorname{Im} \lambda|} \|\mu A^*\| \leq \frac{1}{2}.$$

In it the inequality holds

$$\|(E + \lambda T_\Omega^*)^{-1}\| \leq \frac{|\lambda|}{|\operatorname{Im} \lambda|} \|E + A^*\| \leq \frac{\|E + A^*\|}{2\mu\|A^*\|}.$$

In the angle supplementary to α ,

$$|((E - \lambda T_\Omega^*)^{-1}\varphi, \psi)| \leq e^{M|\lambda|}.$$

Since $((E - \lambda T_\Omega^*)^{-1}\varphi, \psi)$ is an entire function of λ , we may apply to it the Phragmén-Lindelöf theorem on the growth of entire functions. From this theorem it follows that throughout the entire λ -plane the inequality

$$|((E - \lambda T_\Omega^*)^{-1}\varphi, \psi)| < M_1(\varphi, \psi).$$

This means that $(E - \lambda T_\Omega^*)^{-1} = C$, where C is a bounded operator, or $E = (E - \lambda T_\Omega^*)C$, whence $C = E$, $T_\Omega^* = 0$, i.e., for $\varphi \in \Omega$, $T^*\varphi = (E + \mu A^*)^{-1}B\varphi = 0$ for $\varphi \neq 0$. Hence $B\varphi = 0$ for $\varphi \neq 0$, and we arrive at a contradiction with the first condition of the theorem. Thus, $\Omega = 0$, and the system of eigenfunctions and associated functions of equation (5) is complete in \mathfrak{H} .

Corollary. *If in the preceding theorem the requirement that the operator $(E + \mu A)^{-1}$ be bounded is replaced by the requirement that the spectrum of the operator A be discrete, then the theorem remains valid.*

Indeed, taking then λ_0 not belonging to the spectrum, we replace the operator μA by the operator $\mu A + \mu_0 A$, $\mu + \mu_0 = \lambda_0$. The rest of the proof remains unchanged.

Let us apply the lemmas stated above and the theorem to equation (4). Since the operator B_{p+1} is positive, equation (4) has a complete system of eigenvectors and associated vectors. Hence the 2^p -th power of problem (1), (2) has a complete

system of eigenfunctions and associated functions. Applying Lemma 1 and S. L. Sobolev' s embedding theorem ⁽²⁾, we obtain the theorem:

Theorem. *The system of eigenfunctions and associated functions of the Dirichlet problem for the strongly elliptic system (1), (2) has a system of eigenfunctions and associated functions complete in $L^2(D)$ for sufficiently small μ .*

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REFERENCES CITED

- ¹ M. I. Vishik, *Matem. sborn.*, **29**, no. 3 (1951).
- ² S. L. Sobolev, *Some Applications of Functional Analysis in Mathematical Physics*, L., 1950.
- ³ T. Carleman, *Math. Zs.*, **9** (1927).
- ⁴ M. V. Keldysh, *DAN*, **77**, No. 1 (1951).
- ⁵ F. Browder, *Proc. Nat. Acad. Sci. USA*, **38**, 39 (1952).

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