THE RELATION BETWEEN THE VARIATION OF A SET AND THE METRIC PROPERTIES OF COMPLEMENTS

1957

SovietRxiv

Abstract

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MATHEMATICS

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THE RELATION BETWEEN THE VARIATION OF A SET AND THE METRIC PROPERTIES OF COMPLEMENTS

(Presented by Academician A. N. Kolmogorov on 26 XII 1956)

In this note, results obtained earlier ([1]) are refined, expressing the relation between the variation of a set and the metric properties of its complement. It is proved that, if one set approximates another set sufficiently well, then the variations of the first must be large relative to the entropy ([2]) of the second set. Results of this kind lead to some interesting estimates from the theory of tabulation.

Notation: E_n is n-dimensional Euclidean space; $X = X_1, X_2, \dots, X_n$ is an orthonormal basis of this space; $x_1, x_2, \dots, x_n$ are the coordinates of an arbitrary point $x \subset E_n$ with respect to this coordinate system; τ_i^k is a k-dimensional coordinate plane in E_n, $i = 1, 2, \dots, C_n^k$; $\beta_i^{n-k}(z)$ is an $(n-k)$-dimensional plane in E_n, containing the point z from τ_i^k and orthogonal to τ_i^k; w is an n-dimensional closed regular* cube in E_n; $V_0^w(e \cap \beta_i^{n-k}(z))$ is the number of components (zero variation) of the intersection of the set $e \subset E_n$ with the plane $\beta_i^{n-k}(z)$, lying strictly inside the cube w;

$$V_{\tau_i^k}^w(e) = \int_{\tau_i^k} V_0^w(e \cap \beta_i^{n-k}(z))\, dz$$

is the variation of order k of the set e inside the cube w with respect to the plane τ_i^k;

$$V_k^w(e) = \frac{1}{C_n^k} \sum_{i=1}^{C_n^k} V_{\tau_i^k}^w(e)$$

is the variation of order k of the set e inside the cube w;

$$V^w(e) = \sum_{k=0}^{n} V_k^w(e)$$

is the variation of the set e inside the cube w; $A(e, w)$ is the depth of immersion of the set e in the cube w (the upper bound of the distances from the points of the set $e \cap w$ to the boundary of the cube w); $N_\varepsilon(e)$ is the cardinality of a minimal ε-net of the set $e \subset E_n$ in the metric C (as the distance between points one takes the maximum of the absolute coordinate differences); e_i^k is the projection of the set e onto the plane τ_i^k;

$$\mu_\varepsilon^k(e) = \max\left[\frac{\log N_\varepsilon(e_i^k)}{-\log\varepsilon}\right]$$

is the maximum (over i) of the metric ε-dimension of the projection of the set e onto a k-dimensional coordinate plane; in particular,

$$\mu_\varepsilon^n(e) = -\frac{\log N_\varepsilon(e)}{\log\varepsilon}.$$

Lemma 1. Let e be a closed subset of E_n such that for every positive l

$$V_{p+1}^w(e) = 0 \quad (p < n).$$

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* We shall call a cube regular if its edges are parallel to the coordinate axes.
** Here and everywhere below we shall mean the binary logarithm.

Then there exists $k \geqslant 0$ such that

$$V_k^w(e) \geqslant \frac{|A(e, w)|}{2^k C_n^k}.$$

The proof of the lemma is analogous to the proof of Lemma 58 of paper ([1]) (p. 196) and differs from it only in greater laboriousness. This lemma can apparently be improved; namely, in the denominator of the right-hand side of the estimate one may replace 2^k by $p + 1$.

Lemma 2. Let f be an arbitrary bounded subset of E_n; let $e \subset \omega$ be a set satisfying the conditions of Lemma 1 and such that $V_0^w(e \cap \beta_i^{n-k}(z)) \leqslant V_0$ for all possible $k \leqslant p$, i, z (V_0 depends only on e and ω).

Then in $E_n - e$ one can inscribe a regular closed n-dimensional cube w', whose center belongs to the set f, and the length ε of whose edge may be chosen arbitrarily within the limits prescribed by the inequality

$$\left(\frac{1}{6\varepsilon}\right)^{{\mu_{6\varepsilon}}^{n(f)-\mu}{}^{p}_{6\varepsilon}(f)} \geqslant (24)^p C_n^m V_0,$$

where m is the smaller of p and $n/2$.

Proof. In f fix $N_{6\varepsilon}(f)$ points $\{a_i\}$, pairwise separated from one another (in the sense of the metric C) by not less than 3ε. Denote by w_i the regular cube in E_n with center at the point a_i, of side length 3ε, and by w_i' the cube concentric with w_i and of side length ε.

If we assume that each of the cubes $\{w_i'\}$ contains points of e, then it would follow, by Lemma 1, that for some $k \leqslant p$

$$\sum_i V_k^{w_i}(e) \geqslant \left(\frac{N_{6\varepsilon}(f)}{p}\right)\left(\frac{\varepsilon^k}{2^k C_n^k}\right) > \frac{N_{6\varepsilon}(f)\varepsilon^k}{4^p C_n^m},$$

but

$$\sum_i V_k^{w_i}(e) \leqslant V_0(6\varepsilon^k)\left(\frac{1}{6\varepsilon}\right)^{\mu_{6\varepsilon}{}^{k(f)}}.$$

Consequently,

$$\left(\frac{1}{6\varepsilon}\right)^{\mu_{6\varepsilon}{}^{n(f)-\mu^p_{6\varepsilon}(f)}} \leqslant (24)^p C_n^m V_0,$$

which contradicts the statement of the lemma; i.e., Lemma 2 is proved.

Lemma 3. Let f be an arbitrary bounded subset of E_n; let $e \subset \omega$ be a set satisfying the conditions of Lemma 1 and such that $V^w(e) \geqslant 0$.

Then in $E_n - e$ one can inscribe an n-dimensional closed regular cube with center on f, with side of arbitrary length ε satisfying the inequality

$$\varepsilon^p N_{6\varepsilon}(f) \geqslant 4^p C_n^m V^w(e)$$

(m as in the statement of Lemma 2).

The proof is analogous to the proof of Lemma 2.

Theorem 1. Let $e \subset \omega$ satisfy the conditions of Lemma 2. Then in $\omega - e$ one can inscribe a regular closed n-dimensional cube w' with side length

$$d \geqslant \rho / \sqrt[n-p]{c^n V_0},$$

where ρ is the edge length of the cube w; $c \leqslant 12$ is an absolute constant.

Theorem 2. Let $e \subset w$ satisfy the conditions of Lemma 3. Then in $w - e$ one can inscribe a regular n-dimensional closed cube w' with side length

$$d \geq \frac{\rho^{\frac{n}{n-p}}}{\sqrt[n-p]{c^n V^w(e)}},$$

where $c \leq 12$ is an absolute constant; ρ is the length of an edge of the cube w.

The theorems follow easily from Lemmas 2 and 3. For this it is enough to take for f the cube concentric with w and with side length equal, for example, to $\rho/3$. The proof that $c \leq 12$ is obtained directly from Lemma 1.

We now estimate the quantity $V_0^w(e \cap \beta_i^{n-k}(z))$ for certain concrete sets.

Lemma 4. Every level set of a polynomial P_k^n in n variables of degree k, considered only in some n-dimensional closed ball, consists of no more than $(k+1)^n$ components and partitions every ball, and consequently the whole space E_n, also into no more than $(k+1)^n$ parts.

The lemma is proved by methods analogous to those set forth in the work of O. A. Oleinik ([3]). In doing this, as the principal polynomial it is recommended to consider not P_k^n, but

$$P_k^n \left[r^2 - \sum_{i=1}^{n} (x_i - x_i^0)^2 \right],$$

where r and $\{x_i^0\}$ are, respectively, the radius and the coordinates of the center of the ball mentioned in the formulation of the lemma.

Definition. A function $r_q^k(y)$, defined on E_p, will be called **piecewise rational** if it satisfies the following conditions:

A. There exists a finite collection of polynomials defined on E_p,

$$P_1, P_2, \dots, P_l,$$

such that the sum of their degrees does not exceed q, and for every collection of indices $\beta_i = \pm 1$ $(i = 1, 2, \dots, l)$ on the corresponding set

$$e_\beta = e_{\beta_1, \beta_2, \dots, \beta_l} = \prod_{i=1}^{l} e^i_{\beta_i}$$

the function $r_q^k(y)$ coincides with some rational function

$$r_\beta^k(y) = \frac{P_\beta^k(y)}{Q_\beta^k(y)} \quad (\beta = (\beta_1, \beta_2, \dots, \beta_l),\ \beta_i = \pm 1),$$

where $e^i_{\beta_i} = \{P_i \geq 0\}$ for $\beta_i = +1$, and $e^i_{\beta_i} = \{P_i \leq 0\}$ for $\beta_i = -1$; $P_i^k(y)$ and $Q_i^k(y) \neq 0$ are polynomials in p variables of degree not higher than k.

B. $r_q^k(y)$ is everywhere on E_p single-valued and continuous.

In short, the function $r_q^k(y)$ is specified as follows: by the level set of the polynomial

$$P = \prod_{i=1}^{l} P_i$$

the space E_p is divided into 2^l closed regions $\{e_\beta\}$, intersecting pairwise only along their boundaries; on each of these $r_q^k(y)$ is given by its own formula, while on the intersection of every pair of regions the corresponding formulas give coincident values.

We shall call the zero level set of the polynomial P the **barrier** of the function $r_q^k(y)$, and the number q the **order** of the barrier.

Let us note that, by Lemma 4,

$$2^l \leq (q+1)^p.$$

Lemma 5. Let $e \subset E_n$ be the image of a p-dimensional closed ball $\Sigma \subset E_p$ under the transformation given by the equalities

$$t_i = r_{q,i}^k(y) \quad (i = 1, 2, \dots, n),$$

where $\{r_{q,i}^k(y)\}$ are piecewise-rational functions of y with common barrier of order $q > 0$.

Then for every regular cube $w \subset E_n$ and arbitrary l, i, z the inequality holds

$$V_0^w(e \cap \beta_i^{n-l}(z)) \leq 2(q+1)^p(2q+2k+1)^p.$$

Proof. By virtue of the continuity of the transformation under consideration, $V_0^w(e \cap \beta_i^{n-l}(z))$ does not exceed the number $V_0(e_0)$ of components of the zero level set e_0 of the function

$$f_z(y) = \sum_{i=1}^{l} \left[r_{q,i}^k(y) - t_i(z)\right]^2$$

(the summation is over the numbers of all coordinate axes contained in $\tau_i^l \ni z$), considered only on the ball Σ. Denote by e_β^0 and $\overline{e}_\beta^0$ the zero levels of the polynomials

$$P_{2k}^{\beta} = \sum_{i=1}^{l} \left[P_{\beta,i}^{k}(y) - t_i(z) Q_{\beta,i}^{k}(y) \right]^2$$

and

$$P_{(2q+2k)}^{\beta} = P_{2k}^{\beta} + (P)^2$$

(respectively), where $P_{\beta,i}^{k}(y)$ and $Q_{\beta,i}^{k}(y)$ are polynomials representing the function $r_{q,i}^{k} = P_{\beta,i}^{k}(y)/Q_{\beta,i}^{k}(y)$.

Taking Lemma 4 into account, it is not difficult to verify the following inequalities:

$$V_0^w(e \cap \beta_i^{n-l}(z)) \leq V_0(e_0) \leq \sum \left[V_0(e_\beta^0) + V_0(\bar{e}_\beta^0) \right] \leq$$

$$\leq 2^l \left[(2k+1)^p + (2q+2k+1)^p \right] \leq$$

$$\leq 2^l \left[2(2k+2q+1)^p \right] \leq 2(q+1)^p (2q+2k+1)^p.$$

The lemma is proved.

Lemma 6. Let f be an arbitrary subset of the regular cube $\omega \subset E_n$; let e be the image of the space E_p in E_n $(p < n)$ under a transformation given by piecewise-rational functions $t_i = r_{q,i}^{k}(y)$ $(i = 1, 2, \dots, n)$.

Then in $E_n - e$ one can inscribe an n-dimensional regular open cube ω' with center at f and with side length ε, determined by the equation

$$\left(\frac{1}{6\varepsilon} \right)^{\mu_{6\varepsilon}{}^{n}(f) - \mu_{6\varepsilon}^{p}(f)} = 2C_n^m [5(q+1)(q+k+1)]^{2p}.$$

The lemma follows easily from Lemmas 2 and 5.

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Received
25 XII 1956

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Note: Figure translations are in progress. See original paper for figures.