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1957

SovietRxiv

Abstract

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A GENERALIZED PROBLEM ON THE STOCHASTIC MOTION OF A PARTICLE

(Presented by Academician V. A. Fock on 29 X 1956)

The usual formulation of the problem of the stochastic motion of a particle, in which a special role is assigned to time t, is poorly adapted to the theory of relativity and, in particular, to the theory of gravitation*. The natural generalization of the usual problem proposed here removes this deficiency and gives a canonical formulation of problems analogous to the problem of the probability of collision of two particles.

In contrast to the usual case, by the state of a material point we shall mean the totality of its space-time position and velocity. We shall assume that the space F of states of a material point is a simple differentiable manifold. For example, it is such if the space of events is Galilean. The dimension of the space F is equal to 7; however, everywhere where only the fact that it is finite is important, we shall denote it by n, keeping possible generalizations in mind.

The motion of a material point in a given external field of forces is specified by a system of equations:

$$\frac{dx_i}{d\tau} = f_i(x_1, \dots, x_n), \qquad i = 1, 2, \dots, n, \tag{1}$$

where $x_1, \dots, x_n$ are coordinates in F. The collection of functions $\{f_1(x_1, \dots, x_n), \dots, f_n(x_1, \dots, x_n)\}$ forms in F a vector field $f(P)$, $P \in F$. The entire manifold of possible motions of the material point for the given field $f(P)$ is represented by the family of vector lines of this field. On each such line a distance $\tau = \tau(P_1, P_2)$ is specified between any two of its points P_1 and P_2, equal to the proper time of the material point which is necessary for its state P_1 to change into the state P_2, with $\tau(P_1, P_2) = -\tau(P_2, P_1)$. The problem of mechanics is posed as follows: given a field $f(P)$ and the initial state P_0 of the material point, find its state $P = \varphi(\tau; P_0)$ after the lapse of proper time τ. This problem is equivalent to finding the solution of the system (1) with the initial conditions $x_i|_{\tau=0} = x_i^0$, $i = 1, \dots, n$.

We shall investigate the motion of a particle of a definite kind. The particle is formed in some definite state $P_1 \in F$ and during the time of existence $\tau > 0$ "lives through" the set of states $P = \varphi(\eta; P_1)$, $0 < \eta < \tau$, after which it decays in the state $P_2 = \varphi(\tau; P_1)$. This set of states we shall call the life curve of the particle. We shall assume that the particle, starting from an arbitrary state P, decays with probability

$$\frac{\tau}{T(p)} + o(\tau) \left(\lim_{\tau \to 0} \frac{o(\tau)}{\tau} = 0 \right)$$

in one of the states

$$P' = \varphi(\eta; P), \qquad 0 < \eta < \tau.$$

Let there be given a probability $W(D)$ of formation of the particle in a state from the region $D \subset F$, which is determined by a skew-symmetric n-linear—

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* We adhere to the terminology of V. A. Fock (^{1}).

form $^{(2)}Q(P; \xi_1, \dots, \xi_n)$ according to the formula

$$W(D) = \iint_D \cdots \int Q(P; d_1, \dots, d_n).$$

The substantial coefficient of this form is, generally speaking, a generalized function of the coordinates in F. Let S be a piecewise-smooth hypersurface in F such that its common part with any vector line of the field $f(P)$ is either empty, or consists of a single point, or of a single segment.

The probability $w(S)$ that the life curve of the particle intersects the hypersurface S is expressed by the integral

$$w(S) = \int_S \cdots \int \Omega_0(P; d_1, \dots, d_{n-1})$$

over S of the $(n-1)$-linear skew form

$$\Omega_0(P; \xi_1, \dots, \xi_{n-1}) = A_0(P; f(P), \xi_1, \dots, \xi_{n-1}).$$

The substantial coefficient of the n-linear skew form $A_0(P; \xi_1, \dots, \xi_n)$ is equal to the curvilinear integral

$$A^0_{12\dots n}(x_1, \dots, x_n) = \int_0^\infty Q_{12\dots n}(x_1', \dots, x_n') \frac{\partial(x_1', \dots, x_n')}{\partial(x_1, \dots, x_n)} \exp\left[- \int_0^\tau \frac{d\eta}{T(\varphi(-\eta; P))} \right] d\tau, \tag{2}$$

where $x_1, \dots, x_n$ and $x_1', \dots, x_n'$ are the coordinates of the points P and $\varphi(-\tau; P)$.

Having decayed, the particle may be formed again. Let the conditional probability of restoring the particle to a state from the region $D \subset F$, provided that it has decayed in the state P^*, be equal to the integral

$$\iint_D \cdots \int K(P; d_1, \dots, d_n; P^*).$$

The skew form $K(P;\xi_1,\dots,\xi_n;P^*)$ is a scalar function of the point-parameter P^*. The particle may decay and be restored an unlimited number of times. If the process of decay and restoration does not change the quantities $f(P)$, $T(P)$, and $K(P;\xi_1,\dots,\xi_n;P^*)$, then the probability of k-fold restoration is determined by the skew form

$$Q_k(P;\xi_1,\dots,\xi_n) = \iint_F \cdots \int \frac{K(P;\xi_1,\dots,\xi_n;P^*)}{T(P^*)} A_{k-1}(P^*;d_1^*,\dots,d_n^*), \tag{3}$$

where the skew form $A_{k-1}(P^*;d_1^*,\dots,d_n^*)$ determines the probability that the life curve of a particle restored $k-1$ times intersects the hypersurface S. The form $A_{k-1}(P;\xi_1,\dots,\xi_n)$ depends on $Q_{k-1}(P';\xi_1',\dots,\xi_n')$ in the same way as $A_0(P;\xi_1,\dots,\xi_n)$ depends on $Q(P';\xi_1',\dots,\xi_n')$. Hence, from (2) it follows that the skew form

$$A(P;\xi_1,\dots,\xi_n;\lambda) = \sum_{k=0}^{\infty} \lambda^k A_k(P;\xi_1,\dots,\xi_n)$$

satisfies the kinetic equation:

$$\Omega'(P;\xi_1,\dots,\xi_n) + \frac{A(P;\xi_1,\dots,\xi_n)}{T(P)} = \lambda \iint_F \cdots \int \frac{K(P;\xi_1,\dots,\xi_n;P^*)A(P^*;d_1^*,\dots,d_n^*)}{T(P^*)} + Q(P;\xi_1,\dots,\xi_n) \tag{4}$$

with the condition

$$\lim_{\tau\to\infty} \Omega(\varphi(-\tau;P);\xi_1,\dots,\xi_{n-1};\lambda) = 0,$$

$\Omega'(P;\xi,\xi_1,\dots,\xi_{n-1};\lambda)$ being the exterior derivative of the $(n-1)$-linear skew form

$$\Omega(P;\xi_1,\dots,\xi_{n-1};\lambda) = A(P;f(P),\xi_1,\dots,\xi_{n-1};\lambda).$$

The substantial coefficient of the derivative form is

$$\Omega'_{12\dots n}(x_1,\dots,x_n) = \sum_{i=1}^{n} \frac{\partial}{\partial x_i}[A_{12\dots n}(x_1,\dots,x_n) f_i(x_1,\dots,x_n)].$$

The probability of intersection with the hypersurface S by the life curve of the particle, without taking into account how many times the particle decayed and was restored, is determined by the formula

$$\Omega(P;\xi_1,\dots,\xi_{n-1}) = \sum_{k=0}^{\infty} \Omega_k(P;\xi_1,\dots,\xi_{n-1}) = \Omega(P;\xi_1,\dots,\xi_{n-1};1).$$

The usual formulation of the problem is obtained from the one set forth if S is part of the hypersurface $t = \text{const}$. In the formulation of the problem set forth, gravitation is easily taken into account; however, its advantages over the usual one are obvious and in the theory of relativity, since the hypersurface $t = \text{const}$ is not invariant with respect to Lorentz transformations. In the latter case, as coordinates in F one may choose the Galilean coordinates x, y, z, t of the particle and the coordinates u_1, u_2, u_3 of its velocity, equal to the ratios of the spatial components of its momentum to the rest mass. As the volume element in F it is convenient to take the skew form $\varepsilon(P; d_1, \dots, d_7)$, which on the vectors of elementary displacements along the coordinate lines takes the value

$$\varepsilon(P; d_1, \dots, d_7) = dx\, dy\, dz\, dt\, \frac{du_1 du_2 du_3}{u_4}, \qquad u_4 = \sqrt{1 + \frac{u_1^2 + u_2^2 + u_3^2}{c^2}},$$

and as the area element on the hypersurface S, the 6-linear skew form $\varepsilon(P; f(P), d_1, \dots, d_6)$.

Since

$$f(P) = \{u_1, u_2, u_3, u_4, \omega_1(P), \omega_2(P), \omega_3(P)\},$$

the area element on the hypersurface $t = \text{const}$ is equal to

$$\varepsilon(P; f(P), d_1, \dots, d_6) = dx\, dy\, dz\, du_1\, du_2\, du_3.$$

Any 7-linear skew form $B(P; \xi_1, \dots, \xi_7)$ differs from $\varepsilon(P; \xi_1, \dots, \xi_7)$ only by a scalar factor:

$$B(P; \xi_1, \dots, \xi_7) = B(P)\varepsilon(P; \xi_1, \dots, \xi_7).$$

The scalar function $Q(P)$ obtained from this formula is the probability density that, in the state P, the particle under consideration is formed. The scalar function $A_k(P)$ on any hypersurface S is the probability density for the intersection of the world line of the particle, reconstructed k times, with this hypersurface (at the point P). Passing from skew forms to scalar functions, we obtain the kinetic equation in the theory of relativity in the form:

$$u_4 \frac{\partial A(P)}{\partial t} + u_1 \frac{\partial A(P)}{\partial x} + u_2 \frac{\partial A(P)}{\partial y} + u_3 \frac{\partial A(P)}{\partial z} +$$

$$+ u_4 \sum_{l=1}^{3} \frac{\partial}{\partial u_l} \left[\frac{\omega_l(P) A(P)}{u_4} \right] + \frac{A(P)}{T(P)} =$$

$$= \lambda \int_{-\infty}^{+\infty} \int \cdots \int \frac{A(P^*) K(P; P^*)}{T(P^*)}\, dx^*\, dy^*\, dz^*\, dt^*\, \frac{du_1^* du_2^* du_3^*}{u_4^*} + Q(P) \qquad (5)$$

with the condition

$$\lim_{\tau \to +\infty} A(\varphi(-\tau; P)) = 0.$$

Usually

$$T^{-1}(P^*)K(P;P^*) = L(P;a^*)\delta(x-x^*)\delta(y-y^*)\delta(z-z^*)\delta(t-t^*),$$

where a^* is the velocity of the particle in the state P^*. The invariance with respect to Lorentz transformations under this approach to the problem is obvious.

The author is deeply grateful to Acad. V. A. Fock for valuable comments.

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Received
13 X 1956

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Note: Figure translations are in progress. See original paper for figures.