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**Abstract**

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## A LINEAR-GEOMETRIC ANALOG OF A TRI-ORTHOGONAL SYSTEM OF SURFACES

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In his classical memoir ([1]), S. Lie was the first to arrive at a fundamental analogy between the geometry of four-dimensional line space immersed in three-dimensional point space and the so-called higher geometry of spheres. Starting partly from this discovery and partly independently, F. Klein ([2]), in turn, established a suitably understood identity between line geometry and conformal geometry of four-dimensional point space in hexaspherical coordinates. In particular, he succeeded in constructing a quadruple involutive system of complexes as an analog of a quadruple orthogonal system of surfaces in four-dimensional space. A. M. Vasil' ev ([3]) arrived at this same system of complexes by solving directly, by Cartan' s method of exterior forms, the problem of finding it.

In the present paper, in development of the ideas of S. Lie and F. Klein formulated above, a certain parallel is established (although far from a complete one) between triorthogonal systems of surfaces in three-dimensional point space and certain particular complexes of straight lines in four-dimensional line space. We shall arrive at this parallel by solving the following special problem, which nevertheless is of independent interest.

It is known ([2,3]) that each of the complexes belonging to a quadruple involutive system decomposes in three ways into $\infty^1$ congruences $W$, whose foci coincide with the points of contact of the corresponding principal surfaces of the complex. What properties will be possessed by a separately taken complex that decomposes into $\infty^1$ congruences whose foci coincide with the points of contact of one of the principal surfaces of this complex?

By canonizing a tetrahedron in a certain way, we arrive at the following system of equations determining the motion of this tetrahedron ([4]):

$$\omega_{13} + \omega_{24} = 0, \qquad \omega_{31} + \omega_{42} = k\omega_{24},$$

$$\omega_{21} = \frac{\beta}{2(1-\beta^2)} \left[(3-\beta^2)\left(P + \frac{1}{2}\right) + (1+\beta^2)\left(Q - \frac{1}{2}\right)\right]\omega_{23} +$$

$$+ \frac{1}{2(1-\beta^2)} \left[(1+\beta^2)\left(P + \frac{1}{2}\right) - (1-3\beta^2)\left(Q - \frac{1}{2}\right)\right]\omega_{14},$$

$$\omega_{34} = \frac{1}{2(1-\beta^2)}\left[-(1-3\beta^2)\left(P+\frac{1}{2}\right)+(1+\beta^2)\left(Q-\frac{1}{2}\right)\right]\omega_{23}+$$

$$+\frac{\beta}{2(1-\beta^2)}\left[(1+\beta^2)\left(P+\frac{1}{2}\right)+(3-\beta^2)\left(Q-\frac{1}{2}\right)\right]\omega_{14},$$

$$\omega_{12} = \frac{1}{2(1-\beta^2)}\left[-(1-3\beta^2)\left(P-\frac{1}{2}\right)+(1+\beta^2)\left(Q+\frac{1}{2}\right)\right]\omega_{23}+$$

$$+\frac{\beta}{2(1-\beta^2)}\left[(1+\beta^2)\left(P-\frac{1}{2}\right)+(3-\beta^2)\left(Q+\frac{1}{2}\right)\right]\omega_{14}.$$

$$\begin{aligned}\omega_{43} = \frac{\beta}{2(1-\beta^2)}\left[(3-\beta^2)\left(P-\frac{1}{2}\right)+(1+\beta^2)\left(Q+\frac{1}{2}\right)\right]\omega_{23}+\\ +\frac{1}{2(1-\beta^2)}\left[(1+\beta^2)\left(P-\frac{1}{2}\right)-(1-3\beta^2)\left(Q+\frac{1}{2}\right)\right]\omega_{14},\end{aligned} \tag{1}$$

$$\omega_{11} = -\omega_{44} = -\frac{1}{8}(3s+a)\omega_{23} - \frac{1}{8}(3b+r)\omega_{14} + \frac{\beta}{2}(3Q-P)\omega_{24},$$

$$\omega_{22} = -\omega_{33} = -\frac{1}{8}(3a+s)\omega_{23} - \frac{1}{8}(3r+b)\omega_{14} - \frac{\beta}{2}(3P-Q)\omega_{24},$$

$$d\beta = \frac{1}{2}[\beta(a+s)-2r]\omega_{23} + \frac{1}{2}[\beta(b+r)-2s]\omega_{14} + [(\beta^2-1)(P-Q)-k]\omega_{24}.$$

The differential equations of the principal surfaces have the form:

$$\omega_{23} = 0, \qquad \omega_{14} = 0, \tag{2}$$

$$\omega_{23} - \omega_{14} = 0, \qquad \omega_{24} = 0, \tag{3}$$

$$\omega_{23} + \omega_{14} = 0, \qquad \omega_{24} = 0. \tag{4}$$

Geometrically the position of the tetrahedron is determined as follows. The vertices $A_1$ and $A_2$ are placed at the points of contact of the surface (2). The edges $A_1A_3$ and $A_2A_4$ coincide with the generators of the quadric tangent to the surface (2). The edges $A_1A_4$, $A_2A_3$ are polar-conjugate with respect to this quadric and, moreover, are subject to the condition

$$(A_1A_4, A_1A_2, A_1M_3, A_1M_4) = (A_2A_3, A_2A_1, A_2N_3, A_2N_4),$$

where $A_1M_3$, $A_1M_4$, $A_2N_3$, $A_2N_4$ are the corresponding generators of quadrics tangent to the surfaces (3) and (4).

If the ray $A_1A_2$ describes a congruence whose foci are the points $A_1$ and $A_2$, then the equality

$$\omega_{24} = 0 \tag{5}$$

will be the differential equation of such a congruence. It is easy to show that the condition of complete integrability of equation (5)

$$P + Q = 0 \tag{6}$$

will at the same time be the condition that the congruence (5) is a $W$-congruence.

Thus, *if the complex decomposes into $\infty^1$ congruences whose foci coincide with the points of contact of one of the principal surfaces, then these congruences can only be $W$-congruences. Conversely, if the complex contains in each ray a $W$-congruence (the requirement of holonomicity is not imposed), whose foci coincide with the points of contact of a principal surface, then it decomposes into $\infty^1$ such congruences, i.e. the latter automatically turn out to be holonomic.*

The points of contact of the principal surfaces (3) and (4) are respectively

$$M_1 = A_1 + A_2, \quad M_2 = A_1 - A_2$$

and

$$N_1 = A_1 + iA_2, \quad N_2 = A_1 - iA_2$$

($i = \sqrt{-1}$). The differential equations of the congruences whose foci are the points $M_1, M_2$ (respectively $N_1, N_2$) are

$$\omega_{14} + \omega_{23} = 0, \qquad \text{respectively} \quad \omega_{14} - \omega_{23} = 0. \tag{7,8}$$

It is easy to show that equality (6) is a necessary and sufficient condition both for the holonomicity of these congruences and for their being $W$-congruences.

Thus, *if the complex decomposes into $\infty^1$ $W$-congruences whose foci coincide with the points of contact of one of its principal surfaces, then it decomposes in two more ways into $\infty^1$ $W$-congruences whose foci coincide respectively with the points of contact of the two other principal surfaces of the complex. In this case any two principal surfaces of the complex belong to a congruence whose foci coincide with the points*

points of contact of the third principal surface, connecting the asymptotic lines on the focal surfaces of this congruence*.

Thus, we have three two-parameter families of principal surfaces of the complex, which decompose into three one-parameter families of congruences $W$. Taking into account that principal surfaces always intersect one another involutively (see, for example, (5)), which may be regarded as an analogue of orthogonal intersection in conformal geometry, we obtain a certain parallel between the complex under consideration and a triorthogonal system of surfaces in three-dimensional space. This parallel can be described quite satisfactorily by means of the following dictionary:

| | |
|---|---|
| complex (6) | triorthogonal system of surfaces |
| congruences $W$ decomposing the complex | surfaces of the system |
| principal surfaces of the complex | lines of curvature of the surfaces of the system |
| ruled surface of the congruence | line on a surface of the system |
| in particular, a quadric | circle (including a straight line)** |
| developable surface of the congruence | isotropic curve on a surface |

Using this dictionary, one can without much difficulty find line-geometric analogues for certain particular classes of triorthogonal systems of surfaces that can be defined in terms of conformal geometry.

1. **Triorthogonal systems of surfaces, one Lamé family of which is orthogonal to a congruence of straight lines or circles.** In our interpretation this case must correspond to a complex for which all principal surfaces of one family are quadrics. It is easy to show that, if such surfaces are the surfaces (2), then the required complexes will be those for which the equality

$$k = 0. \tag{9}$$

holds identically.

One can calculate that the equalities (1), under the conditions (6) and (9), admit a group of transformations of the tetrahedron

$$\rho_1 \bar{A}_1 = A_1 + \lambda A_3, \quad \rho_2 \bar{A}_2 = A_2 - \lambda A_4, \quad \rho_3 \bar{A}_3 = A_3 + \mu A_1, \quad \rho_4 \bar{A}_4 = A_4 - \mu A_2, \tag{10}$$

where the parameters $\rho_1, \rho_2, \rho_3, \rho_4, \mu$ satisfy certain equalities, and the coefficient $\lambda$ is subject to the following exterior differential equation

$$[d\lambda + \lambda(\omega_{33} - \omega_{11}) - \lambda^2 \omega_{31}, \omega_{24}] = 0. \tag{11}$$

The transformation (10) is nothing other than the *rolling of the congruence* (5), *which decomposes the complex, along the quadric* (2). In the triorthogonal system this corresponds to the displacement of a surface of one Lamé family along the congruence of straight lines or circles orthogonal to it. Relation (11) corresponds, evidently, to the condition of constancy of the distances between two surfaces of the family in the case when they are orthogonal to one and the same congruence of straight lines.

The vertices $\bar{A}_1, \bar{A}_2$ describe the focal surfaces of the congruences $W$ that decompose the complex. The totality of the edges $\bar{A}_1A_3$ and $A_2A_4$ of the tetrahedron in the complex under consideration degenerates into a pair of congruences that decompose in the ordinary sense; moreover, the decomposing surfaces are the indicated focal surfaces of the congruences $W$.

Under the transformation (10) the points of contact of the principal surfaces (3) and (4) move along the generators of the quadric (2). If we denote these

* It is easy to show that the class of complexes determined by equality (6) depends on three functions of two

** It is assumed that the model of conformal space is an adapted Euclidean space.

the generators, respectively, $M_1M_3,\ M_2M_4,\ N_1N_3,\ N_2N_4$, then analogously one can establish that *each pair of congruences* $(M_1M_3,\ M_2M_4)$ and $(N_1N_3,\ N_2N_4)$ *is stratifiable, and their stratifying surfaces will be, respectively, the focal surfaces of the congruences* (7) *and* (8).

**2. Triorthogonal systems of surfaces, one Lamé family of which consists of spheres or planes.** Such systems can be characterized by the property that, in them, *the orthogonal trajectories of the named family are stratified by isotropic curves of the surfaces of this family.*

Using our dictionary, we must, therefore, seek such complexes (6) for which the principal surfaces (2), along the developable surfaces of the congruence (5) (the equations of such surfaces are $\omega_{14} = \omega_{13} = 0$ and $\omega_{23} = \omega_{24} = 0$), are stratified into holonomic congruences. Analytically this means the validity of the identities $[D\omega_{14}, \omega_{14}] = [D\omega_{23}, \omega_{23}] = 0$, which, under condition (6), immediately gives

$$P = Q = 0. \tag{12}$$

Substitution of these values into system (1) leads, in particular, to two possibilities: 1) $h = 0$, 2) $h = \text{const} \neq 0$, where $h$ is the coefficient of one of the equalities (for example $\omega_{32} = h\omega_{14}$) obtained by prolonging system (1).

Investigating case 1), we obtain a class of complexes determined with arbitrariness of one function of two arguments and representing a special case of class (9). *In the complexes under consideration the edge $A_3A_4$ is fixed, while the edges*

*$A_1A_3$, $A_2A_4$ describe two linear congruences having $A_3A_4$ as their common and, moreover, unique directrix. All principal surfaces* (2) *are quadrics containing one and the same generator $A_3A_4$.*

In case 2) we also obtain a certain special class of complexes (9), but now the breadth of the class depends on two functions of one argument.

In accordance with our dictionary, in each case we must assign a certain special class of triorthogonal systems containing one family of spheres with circular or rectilinear orthogonal trajectories passing through one point. To the type of complexes considered one should also assign the complexes of projective rotation, excluded by us in constructing the canonical tetrahedron. These complexes also satisfy the conditions $[D\omega_{14}, \omega_{14}] = [D\omega_{23}, \omega_{23}] = 0$. Each such complex, as is known, can be obtained from an arbitrary congruence $W(\sigma)$ with ruled focal surfaces if in each of its focal planes one draws a pencil of straight lines with center at the point of that focal surface which is not tangent to this plane [6].

As the congruences corresponding to the surfaces (spheres) of the triorthogonal system, we have linear congruences whose directrices are the corresponding rectilinear generators of the focal surfaces of the congruence $\sigma$.

To construct the principal surface which is an analogue of an orthogonal trajectory of the family of spheres, one should take on the two focal surfaces of the congruence $\sigma$ any two curves $S_1$, $S_2$, enveloped by the rays of the congruence. Let $A_1$ be an arbitrary point of the curve $S_1$; let $A_3$ be the corresponding focus on the other focal surface. Suppose the generator of this surface passing through the point $A_3$ intersects the curve $S_2$ at the point $A_2$. Then, as the point $A_1$ moves along the curve $S_1$, the ray $A_1A_2$ will describe the required principal surface.

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*Note: Figure translations are in progress. See original paper for figures.*