A GENERALIZATION OF THE THEORY OF STEADY CURRENTS OF A SHALLOW SEA TO THE CASE OF A VARIABLE COEFFICIENT OF VERTICAL EXCHANGE

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Abstract

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A GENERALIZATION OF THE THEORY OF STEADY CURRENTS OF A SHALLOW SEA TO THE CASE OF A VARIABLE COEFFICIENT OF VERTICAL EXCHANGE

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We shall generalize the theory of steady currents of a shallow sea ([1,2]) to the case when the coefficient of vertical exchange varies both in the horizontal and in the vertical directions.

We write the equations of steady motion along the horizontal Cartesian coordinate axes X, Y in the form

$$\frac{\partial}{\partial z}\left(A\frac{\partial u}{\partial z}\right) = -g\frac{\partial \xi}{\partial x}; \qquad \frac{\partial}{\partial z}\left(A\frac{\partial v}{\partial z}\right) = -g\frac{\partial \xi}{\partial y}, \tag{1}$$

where u, v are the horizontal components of velocity; g is the acceleration of gravity; $\xi(x, y)$ is the lowering of the physical surface of the sea relative to its undisturbed horizontal position $z = 0$ (the Z-axis is directed downward); $A(x, y, z)$ is the kinematic coefficient of vertical exchange.

The boundary conditions on the sea surface, referred approximately to the undisturbed sea surface $z = 0$, and on the bottom $z = H(x, y)$:

for $z = 0$

$$A\frac{\partial u}{\partial z} = -\frac{T_x}{\rho}, \qquad A\frac{\partial v}{\partial z} = -\frac{T_y}{\rho}; \tag{2}$$

for $z = H$

$$u = v = 0, \tag{3}$$

where, in the case of a gently sloping shore, (3) is simultaneously also the condition on the shoreline, where $H = 0$.

Integrating equations (1) under conditions (2) and (3), we obtain

$$u = g\frac{\partial \xi}{\partial x}\int_z^H \frac{\eta}{A}\,d\eta + \frac{T_x}{\rho}\int_z^H \frac{1}{A}\,d\eta;$$

$$v = g\frac{\partial \xi}{\partial y}\int_z^H \frac{\eta}{A}\,d\eta + \frac{T_y}{\rho}\int_z^H \frac{1}{A}\,d\eta. \tag{4}$$

Integrating (4) with respect to z over the limits from $z = 0$ to $z = H$, we obtain the following expressions for the components of the total transport S_x and S_y:

$$S_x = g\frac{\partial \xi}{\partial x}\int_0^H\int_z^H \frac{\eta}{A}\,d\eta\,dz + \frac{T_x}{\rho}\int_0^H\int_z^H \frac{1}{A}\,d\eta\,dz;$$

$$S_y = g\frac{\partial \xi}{\partial y}\int_0^H\int_z^H \frac{\eta}{A}\,d\eta\,dz + \frac{T_y}{\rho}\int_0^H\int_z^H \frac{1}{A}\,d\eta\,dz. \tag{5}$$

Solving (5) with respect to the slopes of the sea surface and introducing the function of total transports $\psi(x, y)$, we obtain

$$\frac{\partial \xi}{\partial x} = \frac{1}{g\displaystyle\int_0^H\int_z^H \frac{\eta}{A}\,d\eta\,dz}\left[-\frac{\partial \psi}{\partial y} - \frac{T_x}{\rho}\int_0^H\int_z^H \frac{1}{A}\,d\eta\,dz\right];$$

$$\frac{\partial \xi}{\partial y} = \frac{1}{g\displaystyle\int_0^H\int_z^H \frac{\eta}{A}\,d\eta\,dz}\left[\frac{\partial \psi}{\partial x} - \frac{T_y}{\rho}\int_0^H\int_z^H \frac{1}{A}\,d\eta\,dz\right]. \tag{6}$$

Substituting (6) into (4), we further obtain

$$u = -\frac{\displaystyle\int_z^H \frac{\eta}{A}\,d\eta}{\displaystyle\int_0^H\int_z^H \frac{\eta}{A}\,d\eta\,dz}\frac{\partial \psi}{\partial y} + \frac{T_x}{\rho}\left[\int_z^H \frac{1}{A}\,d\eta - \frac{\displaystyle\int_0^H\int_z^H \frac{1}{A}\,d\eta\,dz}{\displaystyle\int_0^H\int_z^H \frac{\eta}{A}\,d\eta\,dz}\int_z^H \frac{\eta}{A}\,d\eta\right];$$

$$v = \frac{\displaystyle\int_z^H \frac{\eta}{A}\,d\eta}{\displaystyle\int_0^H\int_z^H \frac{\eta}{A}\,d\eta\,dz}\frac{\partial \psi}{\partial x} + \frac{T_y}{\rho}\left[\int_z^H \frac{1}{A}\,d\eta - \frac{\displaystyle\int_0^H\int_z^H \frac{1}{A}\,d\eta\,dz}{\displaystyle\int_0^H\int_z^H \frac{\eta}{A}\,d\eta\,dz}\int_z^H \frac{\eta}{A}\,d\eta\right]. \tag{7}$$

Thus, $\partial\xi/\partial x$, $\partial\xi/\partial y$ and u, v are expressed through the function ψ, to the determination of which the problem is also reduced. We obtain the equation for ψ, eliminating ξ from (6) by cross differentiation:

$$\frac{\partial}{\partial x}\left[\frac{\dfrac{\partial\psi}{\partial x}}{\displaystyle\int_0^H\int_z^H\frac{\eta}{A}\,d\eta\,dz}\right]+\frac{\partial}{\partial y}\left[\frac{\dfrac{\partial\psi}{\partial y}}{\displaystyle\int_0^H\int_z^H\frac{\eta}{A}\,d\eta\,dz}\right]=$$

$$=\frac{\partial}{\partial x}\left[\frac{\dfrac{T_y}{\rho}\displaystyle\int_0^H\int_z^H\frac{1}{A}\,d\eta\,dz}{\displaystyle\int_0^H\int_z^H\frac{\eta}{A}\,d\eta\,dz}\right]-\frac{\partial}{\partial y}\left[\frac{\dfrac{T_x}{\rho}\displaystyle\int_0^H\int_z^H\frac{1}{A}\,d\eta\,dz}{\displaystyle\int_0^H\int_z^H\frac{\eta}{A}\,d\eta\,dz}\right]. \tag{8}$$

If the sea is closed, then the condition for the function ψ on its contour L may be written in the form

$$(\psi)_L = 0; \tag{9}$$

if there are liquid boundaries, then on them the function ψ must be specified from some additional considerations.

The components of the tangential wind stress entering into (6), (7), and (8) are determined through the modulus W and the components W_x, W_y of the wind velocity by the formulas [2]

$$T_x = \gamma W W_x; \qquad T_y = \gamma W W_y, \tag{10}$$

in which the constant $\gamma = 3.2 \cdot 10^{-6}$ g/cm^3.

Let us assume that the coefficient of vertical exchange depends on the wind speed, the sea depth, and the vertical coordinate. Then from dimensional considerations we shall have [6]

$$A = c\overline{W}H\varphi(\bar{z}), \tag{11}$$

where the constant c and the function φ are dimensionless ($\bar{z} = z/H$), with $\varphi(0) = 1$.

To determine the constant c in formula (11), let us consider the problem of determining the current produced by a uniform wind in an enclosed shallow sea of constant depth. In this case (8) reduces to Laplace' s equation for the function ψ, whose solution is $\psi \equiv 0$. Setting $\psi \equiv 0$ in (7), taking into account (10), (11), and introducing the wind coefficient k, defined as the ratio of the velocity of the surface current in an enclosed sea of constant depth to the velocity of the uniform wind, we obtain

$$c = -\frac{\gamma}{k\rho}\left\{-\frac{\int_0^1 d\bar{z}\int_{\bar{z}}^1 \frac{1}{\varphi}\,d\bar{z}}{\int_0^1 d\bar{z}\int_{\bar{z}}^1 \frac{\bar{z}}{\varphi}\,d\bar{z}}\int_0^1 \frac{\bar{z}}{\varphi}\,d\bar{z} + \int_0^1 \frac{1}{\varphi}\,d\bar{z}\right\}. \tag{12}$$

The function $\varphi(\bar{z})$, characterizing the variation of the coefficient A' in the vertical direction, is usually prescribed in the form of a power law

$$\varphi(\bar{z}) = (1-\bar{z})^n. \tag{13}$$

The case $n = 0$ (the coefficient A does not vary in the vertical direction) was considered by us earlier ([2]). Here we shall consider the cases $n = \frac{1}{2}$ and $n = \frac{3}{4}$, in accordance with the results obtained by Sverdrup ([3]) and Fjeldstad ([4]) on the basis of observations of currents in the East Siberian Sea.

Formula (12), for n equal to 0, $\frac{1}{2}$, and $\frac{3}{4}$, gives values of the constant c equal, respectively, to $\gamma/4k\rho$, $\gamma/3k\rho$, and $2\gamma/5k\rho$. For the indicated values, from (6), (7), (8) one can obtain the following final expressions for the slopes of the sea surface and the horizontal components of the current velocity, as well as the equations for the function of the total transports:

1. For $n = 0$

$$\frac{\partial\xi}{\partial x} = -\frac{3\gamma W}{4g\rho k H^2}\frac{\partial\psi}{\partial y} - \frac{3\gamma W W_x}{2g\rho H}, \qquad \frac{\partial\xi}{\partial y} = \frac{3\gamma W}{4g\rho k H^2}\frac{\partial\psi}{\partial x} - \frac{3\gamma W W_y}{2g\rho H}; \tag{14}$$

$$u = -\frac{3(1-\bar{z}^2)}{2H}\frac{\partial\psi}{\partial y} + kW_x(1-4\bar{z}+3\bar{z}^2),$$

$$v = \frac{3(1-\bar{z}^2)}{2H}\frac{\partial\psi}{\partial x} + kW_y(1-4\bar{z}+3\bar{z}^2); \tag{15}$$

$$\frac{\partial}{\partial x}\left(\frac{W}{H^2}\frac{\partial\psi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{W}{H^2}\frac{\partial\psi}{\partial y}\right) = 2k\,\mathrm{rot}_z\,\frac{W\mathbf{W}}{H}, \tag{16}$$

where

$$\mathrm{rot}_z\,\frac{W\mathbf{W}}{H} = \frac{\partial}{\partial x}\left(\frac{WW_y}{H}\right) - \frac{\partial}{\partial y}\left(\frac{WW_x}{H}\right). \tag{17}$$

2. For $n = \frac{1}{2}$ *

$$\frac{\partial \xi}{\partial x} = -\frac{5\gamma W}{16 g\rho k H^2}\frac{\partial \psi}{\partial y} - \frac{5\gamma W W_x}{4 g\rho H},$$

$$\frac{\partial \xi}{\partial y} = \frac{5\gamma W}{16 g\rho k H^2}\frac{\partial \psi}{\partial x} - \frac{5\gamma W W_y}{4 g\rho H}; \tag{18}$$

* The solution of the simpler problem of a current produced by a uniform wind in an enclosed sea of constant depth is due to V. B. Shtokman ([5]). He assumed that the coefficient of vertical exchange does not depend on the horizontal coordinates.

$$u = -\frac{5\sqrt{1-\bar{z}}\,(2+\bar{z})}{8H}\frac{\partial \psi}{\partial y} + kW_x\sqrt{1-\bar{z}}\left(1 - \frac{5\bar{z}}{2}\right), \tag{19}$$

$$v = \frac{5\sqrt{1-\bar{z}}\,(2+\bar{z})}{8H}\frac{\partial \psi}{\partial x} + kW_y\sqrt{1-\bar{z}}\left(1 - \frac{5\bar{z}}{2}\right);$$

$$\frac{\partial}{\partial x}\left(\frac{W}{H^2}\frac{\partial \psi}{\partial x}\right) + \frac{\partial}{\partial x}\left(\frac{W}{H^2}\frac{\partial \psi}{\partial y}\right) = 4k \operatorname{rot}_z \frac{W\mathbf{W}}{H}. \tag{20}$$

3. For $n = 3/4$

$$\frac{\partial \xi}{\partial x} = -\frac{9\gamma W}{64 g\rho k H^2}\frac{\partial \psi}{\partial y} - \frac{9\gamma W W_x}{8 g\rho H}, \tag{21}$$

$$\frac{\partial \xi}{\partial x} = \frac{9\gamma W}{64 g\rho k H^2}\frac{\partial \psi}{\partial x} - \frac{9\gamma W W_y}{8 g\rho H};$$

$$u = -\frac{9\sqrt[4]{1-\bar{z}}\,(4+\bar{z})}{32H}\frac{\partial \psi}{\partial y} + kW_x\sqrt[4]{1-\bar{z}}\left(1 - \frac{9\bar{z}}{4}\right), \tag{22}$$

$$v = \frac{9\sqrt[4]{1-\bar{z}}\,(4+\bar{z})}{32H}\frac{\partial \psi}{\partial x} + kW_y\sqrt[4]{1-\bar{z}}\left(1 - \frac{9\bar{z}}{4}\right);$$

$$\frac{\partial}{\partial x}\left(\frac{W}{H^2}\frac{\partial \psi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{W}{H^2}\frac{\partial \psi}{\partial y}\right) = 8k \operatorname{rot}_z \frac{W\mathbf{W}}{H}. \tag{23}$$

From equations (16), (20), and (23) it is seen that the qualitative picture of water exchange, characterized by the direction of the total transports, remains the same in all three cases. Quantitatively, however, the water exchange changes quite substantially: its intensity increases from the first to the second and third cases, respectively, by two and four times.

As for the slopes of the sea surface, in contrast to the water exchange, they change only slightly. Indeed, from formulas (14), (18), and (21) there follows a simple dependence between the slopes of the sea surface in the cases considered: in passing from the first case to the second and third, they decrease, respectively, by 1/6 and 1/4, i.e., by only 17 and 25%.

Regarding the current velocities, the following may be noted. Curves of the vertical distribution of velocities constructed with the coefficient A varying in the vertical direction are closer to the corresponding curves constructed on the basis of direct observations of currents in the sea. In practical calculations one may recommend a computational scheme constructed either for $n = 1/2$ or for $n = 3/4$. A comparison of the computed and observed current velocities will show the advantage of one scheme or the other.

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Note: Figure translations are in progress. See original paper for figures.