ON THE UNIFORM CONVERGENCE OF EXPANSIONS IN EIGENFUNCTIONS OF ODD-DIMENSIONAL DOMAINS

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Abstract

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ON THE UNIFORM CONVERGENCE OF EXPANSIONS IN EIGENFUNCTIONS OF ODD-DIMENSIONAL DOMAINS

(Presented by Academician S. L. Sobolev on 7 III 1957)

In the present paper we study the question of the uniform convergence of expansions in eigenfunctions of the equation $\Delta u + \lambda u = 0$ in an arbitrary domain g of any odd number of dimensions N, with a homogeneous boundary condition of any of the three types, under the condition that summation is carried out in the natural order of increasing eigenvalues.

For an arbitrary domain of any even number of dimensions, the corresponding question was studied by us in paper (^{1}). The abandonment of the requirement of absolute convergence of the Fourier series and the study of the uniform convergence of this series when summed in the natural order of increasing eigenvalues made it possible for us, in the cited paper, substantially to weaken the usual conditions of expandability.

In particular, in (^{1}) the following assertion was proved:
Let g be an arbitrary domain of any even number N of dimensions, admitting the application of Green' s formulas to eigenfunctions, and let f be an arbitrary function defined in this domain and satisfying the following two requirements:

1) $f \in W_p^{(N/2)}(g)$, where $p > 2$;
2) the function f itself, its Laplacian Δf, and its iterated Laplacians up to order* $[(N-2)/4]$ in the case of the first boundary-value problem and up to order $[(N-4)/4]$ in the case of the second or third boundary-value problem** satisfy, in the generalized sense (i.e., "on the average"), the corresponding homogeneous boundary condition.

Then the Fourier series of the function f converges, when summed in the order of increasing eigenvalues, uniformly in any strictly interior subdomain g'.

In (^{1}) it was indicated that the smoothness condition found, $f \in W_p^{(N/2)}$, where $p > 2$, is, for an arbitrary even-dimensional domain, sharp and cannot be improved (see Remark 3).

In the present paper we prove the corresponding expansion theorem for an arbitrary odd-dimensional domain and also find, for this domain, the sharp smoothness condition:

$$f \in W_p^{((N-1)/2)}(g), \quad \text{where } p > \frac{2N}{N-1}.$$

Comparison of the sharp smoothness requirements for domains of even and odd numbers of dimensions leads us to the conclusion that the expandability conditions for even and odd numbers of dimensions are different.

We proceed to a detailed formulation of the results.

Main theorem. *Let g be an arbitrary domain of any*** *odd number N of dimensions, admitting the application of Green' s formulas to eigen—*

eigenfunctions, and f is an arbitrary function defined in this domain and satisfying the following two requirements:

1) $f \in W_p^{((N-1)/2)}(g)$, where $p > \dfrac{2N}{N-1}$;
2) the function f itself, its Laplacian Δf, and its iterated Laplacians up to order $[(N-2)/4]$ in the case of the first boundary-value problem, and up to order $[(N-4)/4]$ in the case of the second or third boundary-value problem* satisfy, in the generalized sense (i.e. "in the mean"), the corresponding homogeneous boundary condition.

Then, when summed in the order of increasing eigenvalues, the Fourier series of the function f converges to this function uniformly in any strictly interior subdomain g'.

Remark 1. In "classical" terms, the two requirements imposed on the function f may be formulated as follows:

1) f and its derivatives up to order $(N-3)/2$ are continuous in the closed domain g, and the derivatives of order $(N-1)/2$ are integrable over the domain g with exponent p, where $p > 2N/(N-1)$;
2) f and its iterated Laplacians indicated in requirement 2) satisfy the corresponding boundary condition in the "classical" sense.

Remark 2. Let us emphasize that, in the case where g is a domain of star-shaped type, from the known embedding theorems (^{2}) and from the requirement $f \in W_p^{((N-1)/2)}(g)$, where $p > 2N/(N-1)$, it follows automatically that: 1) f

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* Square brackets here and below mean that the integer part of the number enclosed in them is to be taken.

** For $N = 2$, in the case of the second or third boundary-value problem, satisfaction of the boundary condition is not required at all.

*** Thus the one-dimensional case $N = 1$ is automatically excluded.

is continuous in the closed domain g; 2) f and its iterated Laplacians up to the order indicated in requirement 2 (and, in the case of the second or third boundary-value problem, also the normal derivatives of these Laplacians) are summable with square over any $(N-1)$-dimensional piecewise-smooth manifold. Hence it follows that satisfaction of the boundary conditions contained in requirement 2) does not require the function f to satisfy any additional smoothness requirements besides the fact that $f \in W_p^{((N-1)/2)}$, where $p > 2N/(N-1)$.

Remark 3. We note that the principal smoothness requirement established in the present work,

$$f \in W_p^{((N-1)/2)}, \quad \text{where } p > 2N/(N-1),$$

is sharp: in this requirement one cannot decrease either the order of differentiability $(N-1)/2$ or the summability exponent p.

Indeed, already the case $N = 1$ shows that the order of differentiability $(N-1)/2$ cannot be replaced by $(N-3)/2$ (whatever p! may be), since then the function f being expanded is not, generally speaking, continuous (this function belongs only to L_p).

Next suppose that in the requirement $f \in W_p^{((N-1)/2)}$ the number $p = 2N/(N-1)$. In the work of S. L. Sobolev (^{3}) examples are indicated showing that, if $f \in W_p^{(l)}$ and $pl = N$, then the function f is not, generally speaking, continuous. An example of such a function may be

$$f(Q) = \log\log r_{PQ},$$

where P is a fixed interior point of the N-dimensional domain g, having diameter $d < 1$. It is not difficult to verify that

$$f(Q) \in W_{2N/(N-1)}^{((N-1)/2)}(g).$$

This example shows that the requirement $f \in W_p^{((N-1)/2)}$, where $p > 2N/(N-1)$, is sharp and cannot be improved. Still simpler examples are constructed (an N-dimensional sphere with radial symmetry) indicating that if the function being expanded is arbitrarily smooth but does not satisfy at least one of the boundary conditions contained in requirement 2), then there can be no question of convergence of the Fourier series of this function (at interior points of the domain).

Thus, the requirements imposed on the function f in the formulation of the main theorem, taken together, constitute sharp conditions for expandability.

* For $N = 3$, in the case of the second or third boundary-value problem, satisfaction of the boundary condition is not required at all.

Remark 4. For an arbitrary three-dimensional domain g, the main theorem implies the expandability of a function without the assumption of the existence of its second derivatives.

It suffices to require that the function f possess generalized first derivatives summable with exponent $d > 3$, and, in the case of the first boundary-value problem, satisfy "on the average" a homogeneous boundary condition of the first kind (in the case of the second or third boundary-value problem, satisfaction of a boundary condition is not required at all!).

We outline the proof of the main theorem.

1. First of all, one establishes the initial asymptotic formula of the form

$$\sum_{\sqrt{\lambda_i}<\mu} f_i u_i(Q) = \frac{2}{\sqrt{\pi}} \frac{1}{2^{(N-1)/2}\Gamma(N/2)} \int_0^R \frac{\sin \mu r}{r} D^{(N-1)/2} F_Q(r)\, dr + \alpha(\mu). \quad (1)$$

Here $\alpha(\mu)$ denotes an infinitesimal quantity as $\mu \to \infty$, and the estimate of the α-terms is uniform with respect to Q, provided that Q belongs to an arbitrary strictly interior subdomain g'; R is any number not exceeding the minimum distance between the boundaries of g and g'; $F_Q(r) = r^{N-1}\overline{f}_Q(r)$, where $\overline{f}_Q(r)$ is the "mean" value of f on the surface of the N-dimensional sphere C_r^Q of radius r with center at the point Q, i.e.

$$\overline{f}_Q(r) = \frac{1}{\omega_N r^{N-1}} \iint \cdots \int f(S)\, dS; \qquad \omega_N = \frac{2(\sqrt{\pi})^N}{\Gamma(N/2)}$$

is the area of the surface of the N-dimensional sphere of unit radius; the symbol D denotes the operation

$$DF_Q(r) = \frac{d}{dr}\left[\frac{1}{r}F_Q(r)\right];$$

the symbol D^k denotes the k-fold repeated application of the operation D.

For the derivation of the initial asymptotic formula (1), a scheme is used which has much in common with the scheme for deriving the asymptotic formula set forth by us in papers (^{1,4}). However, a number of entirely new points arise here, connected with estimates of the behavior of generalized derivatives of functions of one variable $F_Q(r)$ and with the convergence of certain auxiliary series. In view of the brevity of the present article, we are forced to restrict ourselves to this indication.

2. Starting from the properties of one-dimensional trigonometric Fourier series, the principal term standing on the right-hand side of formula (1) is investigated, and it is proved that this term has the asymptotic order

$$f(Q) + \alpha(\mu), \tag{2}$$

where $\alpha(\mu)$ is infinitesimal, uniformly with respect to $Q \in g'$. As a result, we arrive at the main asymptotic formula of the form

$$\sum_{\sqrt{\lambda_i} < \mu} f_i u_i(Q) = f(Q) + \alpha(\mu), \tag{3}$$

where $\alpha(\mu)$ is infinitesimal, uniformly with respect to $Q \in g'$. From the asymptotic formula (1) there follows directly the assertion of the main theorem on the uniform convergence of the Fourier series of the function in any strictly interior subdomain g'.

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Note: Figure translations are in progress. See original paper for figures.

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