Soviet-era science, translated into English

MATHEMATICS

V. P. MIKHAILOV

1957

SovietRxiv

View the original and related papers at https://sovietrxiv.org/items/ru-195701.72544

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

SovietRxiv

Abstract

Full Text

MATHEMATICS

V. P. MIKHAILOV

ON THE ANALYTIC SOLUTION OF THE GOURSAT PROBLEM FOR A SYSTEM OF DIFFERENTIAL EQUATIONS

(Presented by Academician I. G. Petrovsky, 14 II 1957)

In the present note the following problem is considered: to find, in some domain $|x| < r_0,\ |t| < r_0$, an analytic solution $u_1(x,t), \dots, u_n(x,t)$ of the system of differential equations

$$F_i(x, t, u_1, \dots, u_n, \partial u_1/\partial x, \dots, \partial u_n/\partial t), \quad i = 1, 2, \dots, n, \tag{1}$$

such that

$$u_i(x,t)|_{l_i} = 0, \quad i = 1, 2, \dots, n, \quad l_i \text{ is the line } x = \mu_i t. \tag{2}$$

If $\mu_i \neq 0,\ i = 1, 2, \dots, n$, then, substituting

$$\frac{\partial u_i}{\partial x} = \sqrt{\frac{1+\mu_i^2}{\mu_i}}\,\frac{\partial u_i}{\partial l_i} - \frac{1}{\mu_i}\frac{\partial u_i}{\partial t}$$

in (1) for $x = t = 0$, we obtain, in view of the fact that $u_i|_0 = \partial u_i/\partial l_i|_0 = 0$, a system of equations with respect to $(\partial u_i/\partial t)_0,\ i = 1, 2, \dots, n$, which in what follows we shall assume to be uniquely solvable. With respect to the functions $F_i(x, t, u_k, \partial u_k/\partial x, \partial u_k/\partial t)$, analyticity is assumed in the domain $|x| < R_0,\ |t| < R_0,\ |u_i| < R_0,\ |\partial u_i/\partial x - (\partial u_i/\partial x)_0| < R_0,\ |\partial u_i/\partial t - (\partial u_i/\partial t)_0| < R_0$, where $(\partial u_i/\partial x)_0 = (-1/\mu_i)(\partial u_i/\partial t)_0$.

Analogously, the case is considered when the functions $u_i(x,t),\ i = 1, 2, \dots, n$, are prescribed not on the lines $l_i,\ i = 1, 2, \dots, n$, but on arbitrary analytic curves passing through the origin of coordinates.

This problem is a direct generalization of problems considered by É. Goursat (1), N. M. Günter (2), and S. L. Sobolev (3).

1. Expanding $F_i(x, t, u_k, \partial u_k/\partial x, \partial u_k/\partial t)$ in a Taylor series in $\partial u_k/\partial x$ and $\partial u_k/\partial t$ in neighborhoods of $(\partial u_k/\partial x)_0$ and $(\partial u_k/\partial t)_0$, respectively, we obtain

SovietRxiv

$$\frac{\partial u_k}{\partial t} = \sum_{j=1}^{n} b_{kj} \frac{\partial u_j}{\partial x} + \Phi_k \left(x, t, \dots, \frac{\partial u_n}{\partial t} \right), \quad k = 1, 2, \dots, n; \tag{3}$$

$$\|b_{ij}\| = \left\| \frac{\partial F_i}{\partial(\partial u_s/\partial t)} \right\|^{-1} \left\| \frac{\partial F_i}{\partial(\partial u_s/\partial x)} \right\|; \quad \left. \frac{\partial \Phi_k}{\partial(\partial u_s/\partial t)} \right|_0 = \left. \frac{\partial \Phi_k}{\partial(\partial u_s/\partial x)} \right|_0 = 0, \tag{4}$$

b_{ij} are real.

Suppose, for simplicity, that the invariant factors of the matrix $\|b_{ij}\| - \lambda E$ are of the first order; let $\lambda_1, \dots, \lambda_n$ be the roots of the equation

$$\det(\|b_{ij}\| - \lambda E) = 0 \tag{5}$$

(the general case of the matrix $\|b_{ij}\|$ does not require an essential change of the method, and all the results of this note are completely carried over to this case as well). Let $S = \|s_{ij}\|$, $\det \|s_{ij}\| \neq 0$, $S^{-1} = \|\sigma_{ij}\|$, $S^{-1} \|b_{ij}\| S = \Lambda =$

$$= \left\| \begin{matrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{matrix} \right\|.$$

The solution of problem (1)–(2) is equivalent to solving, in analytic functions $u_i(x, t)$, $\psi_i(t)$, $i = 1, 2, \dots, n$, the system

$$u_i(x,t) = \sum_{j=1}^{n} s_{ij} \psi_j(x + \lambda_j t) + + \sum_{j=1}^{n} \sum_{j_1=1}^{n} s_{ij} \sigma_{j_1} \int_0^t \Phi_{j_1}(x + (t-\tau)\lambda_{j_1}, \tau, u_r(x + (t-\tau)\lambda_{j_1}, \tau), \dots)\, d\tau; \tag{6}$$

$$0 = \sum_{j=1}^{n} s_{ij} \psi_j(\lambda_{ij} t) + + \sum_{j=1}^{n} \sum_{j_1=1}^{n} s_{ij} \sigma_{j_1} \int_0^t \Phi_{j_1}(\lambda_{ij_1} t - \lambda_{j_1} \tau, \tau, u_r(\lambda_{ij_1} t - \lambda_{j_1} \tau, \tau), \dots)\, d\tau, \tag{7}$$

where $\lambda_{ij} = \lambda_j + \mu_i$, $i, j = 1, 2, \dots, n$.

2. Let us consider the case when the roots $\lambda_1, \dots, \lambda_n$ of equation (5) are real. Then the following theorem is valid.

Theorem 1. *In order that the regular Goursat problem (1)–(2) (the definition of regularity is given below) have a unique solution in the class of analytic functions, it is necessary and sufficient that the point $\mu = (\mu_1, \dots, \mu_n)$ in the n-dimensional space of the angular coefficients of the straight lines (2) lie on none of a finite number of algebraic surfaces*

$$\Delta_k(\mu_1, \dots, \mu_n) = 0, \qquad k = 0, 1, \dots, m_0, \quad m_0 \geqslant 1, \tag{8}$$

which are completely determined by the numbers b_{ij}, $i, j = 1, 2, \dots, n$.

SovietRxiv

Proof. Let us first consider the case when Φ_i do not depend on u_k, $\partial u_k/\partial x$, $\partial u_k/\partial t$, $k = 1, 2, \dots, n$. Then the problem is solved directly: we differentiate (7) m times with respect to t, put $t = 0$, and determine $\psi_i^{(m)}(0)$ uniquely from the resulting system of linear equations, provided only that $\Delta_m(\mu) = \det \|s_{ij}\lambda_{ij}^m\| \neq 0$. In this case

$$\psi_i^{(m)}(0) = \sum_{s=1}^{n} \Delta_{si}^{(m)}(\mu) \sum_{k=1}^{n} \sum_{j=1}^{n} s_{ik}\sigma_{kj} \left(\frac{d^m}{dt^m} \int_0^t \Phi_j(\ )\, d\tau \right)_0 \Big/ \Delta_m(\mu), \qquad (9)$$

where $\Delta_{sj}^{(m)}(\mu)$ is the algebraic complement of the element $s_{sj}\lambda_{sj}^m$ in $\Delta_m(\mu)$. Let

$$\Delta_m(\mu) = \sum_{i_k, j_k} A_{i_1 \dots i_n}^{j_1 \dots j_n} (\lambda_{i_1 j_1} \cdots \lambda_{i_n j} n)^m.$$

After collecting like terms (terms in which the moduli of the expressions $(\lambda_{i_1 j_1} \cdots \lambda_{i_n j} n)^m$ coincide), we obtain

$$\Delta_m(\mu) = \sum_{i_k, j_k} B_{i_1 \dots i_n}^{j_1 \dots j_n}(m) \left| \lambda_{i_1 j_1} \cdots \lambda_{i_n j} n \right|^m .$$

Because the λ_i are real, $B_{i_1 \dots i_n}^{j_1 \dots j_n}(m)$ assumes only two values, depending on the parity of m. Therefore, if $\Delta_0(\mu) \neq 0$, $\Delta_1(\mu) \neq 0$, then for large $m > m_0$

$$|\Delta_m(\mu)| > C[\rho(\mu)]^m,$$

where

$$\rho(\mu) = \min |\lambda_{i_1 j_1} \cdots \lambda_{i_n j} n| > 0, \qquad \lambda_{ij} \neq 0.$$

Hence it is obvious that the conditions $\Delta_m(\mu) \neq 0$ for $m = 0, 1, \dots$ can be replaced by the conditions $\Delta_m(\mu) \neq 0$ for $m = 0, 1, \dots, m_0,\ m_0 \geqslant 1$.

From (8) we have

$$\left| \psi_i^{(m)}(0) \right| \leq \frac{s\sigma n^3 M_1 R_1^m M}{c[\rho(\mu)]^m R^m}\, m! = C_1 M \left(r_1(\mu) \right)^m m!, \qquad (10)$$

where

$$s = \max_{i,j} |s_{ij}|, \qquad \sigma = \max_{i,j} |\sigma_{ij}|, \qquad \left| \Delta_{si}^{(m)}(\mu) \right| \leq M_1 R_1^m, \qquad M = \max_i |\Phi_i|,$$

$$R_1 = \max_{i,j} \left(\max_k \left| \lambda_{i_1 j_1} \cdots \lambda_{i_{k-1} j_{k-1}} \lambda_{i_{k+1} j_{k+1}} \cdots \lambda_{i_n j} n \right| \right), \qquad \left| \frac{d^m}{dt^m} \int_0^t \Phi_i()\, d\tau \right| < \frac{M}{R^m}.$$

SovietRxiv

Estimate (10) proves the analyticity of $\psi_i(t)$ for $|t| < r_1(\mu)$. If in the μ-space one takes a closed domain F containing no points of the surfaces (8), and denotes $r_0 = \inf_{\mu\in F} r_1(\mu) > 0$, then in the domain $|t| < r_0$, for $\mu \in F$, analytic $\psi_i(t)$ exist, and hence, by (6), so does a solution of the Goursat problem (1)–(2). Let now Φ_j depend on $u_k,\ \partial u_k/\partial x,\ \partial u_k/\partial t$. Using (4), it is easy to obtain, in the case when the conditions of Theorem 1 are satisfied, the uniqueness of the solution.

To prove the convergence of the Taylor series for $u_i(x,t)$, $i = 1, 2, \dots, n$, one may use the method of successive approximations. Let $u_i^{(0)} \equiv 0$, $i = 1, 2, \dots, n$, in some bounded domain G_0,

$$u_i^{(m)}(x,t) = \sum_{j=1}^{n} s_{ij}\psi_j^{(m)}(x+\lambda_j t) +$$

$$+ \sum_{j=1}^{n}\sum_{k=1}^{n} s_{ij}\sigma_{jk}\int_0^t \Phi_k(\dots, u_r^{(m-1)}(x+(t-\tau)\lambda_k, \tau), \dots)\, d\tau; \tag{11}$$

$$0 = \sum_{j=1}^{n} s_{ij}\psi_j^{(m)}(\lambda_{ij}t) + \sum_{j=1}^{n}\sum_{k=1}^{n} s_{ij}\sigma_{jk}\int_0^t \Phi_k(\dots, u_r^{(m-1)}(\lambda_{ik}t-\lambda_k\tau, \tau), \dots)\, d\tau. \tag{12}$$

Denote by $G_1(G_0)$ the domain of definition of $u_i^{(1)}(x,t)$, $i = 1, \dots, n$. We shall call problem (1)–(2) **regular** if $G_1(G_0) \supset G_0$.

From (11), (12), taking into account the case considered above when the Φ_i depend only on x and t, there follows the existence of all $u_i^{(m)}(x,t)$, $i = 1, 2, \dots, n$, $m \geq 0$. It remains to prove that: 1) the domains of convergence of $u_i^{(m)}(x,t)$ can be chosen independent of m; 2) the successive approximations do not leave the domain of definition of $\Phi_i(x, t, u_k, \partial u_k/\partial x, \partial u_k/\partial t)$; 3) there exist uniform limits

$$\lim_{m\to\infty} u_i^{(m)}(x,t) = u_i(x,t), \quad i = 1, 2, \dots, n,$$

for $|t| < \rho_0,\ |x| < \rho_0$.

In the general case, the proof of these properties is based mainly on the following auxiliary propositions: 1) if for some point $\mu^0 = (\mu_1^0, \dots, \mu_n^0)$ in the space of angular coefficients of the straight lines (2) there exists a unique analytic solution of the Goursat problem, then this is also true in a small neighborhood of μ^0; 2) one can find an affine transformation (x,t) such that all μ_i pass into μ_i', $\mu_i' < 0$, $|\mu_i'| < \varepsilon$, and λ_i into λ_i', $\lambda_i' > 0$, $\lambda_i' < \varepsilon$, where $\varepsilon > 0$ is arbitrary.

Consider a special case of the system (3), in which properties 1)–3) are easily proved. Let $\rho_i = \max_j |\lambda_{ij}|$, $1 \leq j \leq n$, and suppose that: a) the element

SovietRxiv

$s_{i,j(i)}\lambda_{i,j(i)}$ of the matrix $\|s_{ij}\lambda_{ij}\|$ is the only one in the i-th row with the property

$$\max_j |\lambda_{ij}| = \rho_i = |\lambda_{i,j(i)}|;$$

b) $j(i_1) \neq j(i_2)$ when $i_1 \neq i_2$; c) $s_{i_0 j(i_0)} \neq 0$ if $\lambda_{i_0 j(i_0)} = \rho_{i_0}$. Suppose that for $m = 1$ the first approximation $\psi_i^{(1)}(t)$ is defined for $|t| < r_1$; when properties a), b), c) are satisfied, system (12) permits

analytically continue $\psi_i^{(1)}(t)$ into the disk of the maximum radius allowed by the functions Φ_i, where in advance one may assume that such estimates hold for $\psi_i^{(1)}(t)$ and $u_i^{(1)}(x,t)$ that $u_i^{(1)}(x,t)$ do not leave the domain of definition of Φ_i, etc. The assumption of regularity of the problem, as the example of Méray (5) shows, is essential.

3. Let several roots of equation (5) be nonreal. Then the following theorems are valid (by M is denoted the set in the n^2-dimensional space $\{b_{ij},\ i,j = 1,2,\dots,n\}$ on which among the $\operatorname{Im}\lambda_i$ there are nonzero ones).

Theorem 2. *If among the roots of equation (5) there are nonreal ones, then for some everywhere dense set in M and almost all $\mu(\mu_1,\dots,\mu_n)$ in the n-dimensional space of angular coefficients of the straight lines (2), the Goursat problem has no analytic solution.*

Theorem 3. *If among the roots of equation (5) there are nonreal ones, then for some everywhere dense set in M there exists an everywhere dense set of values $\mu(\mu_1,\dots,\mu_n)$ for which the Goursat problem has a unique analytic solution.*

As a simple example, let us consider the Cauchy–Riemann system of equations

$$\partial u/\partial t = \partial v/\partial x, \qquad \partial u/\partial x = -\partial v/\partial t,$$

and, instead of (2),

$$u(x,t)\big|_{x=\mu t} = \varphi_1(t), \qquad v(x,t)\big|_{x=\nu t} = \varphi_2(t).$$

The general solution of this system can be written in the form

$$u(x,t) = -f_1(x+it) - f_2(x-it), \qquad v(x,t) = -if_1(x+it) + if_2(x-it),$$

and to determine f_i we obtain the system

$$f_1(t) + f_2((\mu-i)t/(\mu+i)) = -\varphi_1(t/(\mu+i)),$$

$$f_1(t) - f_2((\nu-i)t/(\nu+i)) = i\varphi_2(t/(\nu+i))$$

or

$$f_2(t) + f_2(e^{2\pi i\rho}t) = \Phi(t),$$

where

$$\Phi(t) = -\varphi_1(t/(\mu-i)) - i\varphi_2(t(\mu+i)/(\nu+i)(\mu-i)),$$

SovietRxiv

$$e^{2\pi i\rho} = (\nu - i)(\mu + i)/(\nu + i)(\mu - i), \qquad 0 \leqslant \rho \leqslant 1.$$

Differentiating this relation m times with respect to t and putting $t = 0$, we obtain

$$(1 + e^{2\pi i m\rho}) f_2^{(m)}(0) = \Phi^{(m)}(0).$$

It can be shown that if $\rho = p/q$, $(p, q) = 1$, $p \equiv 1 \pmod 2$, $q \equiv 0 \pmod 2$, then this equation has no solution for $m \equiv 0 \pmod{q/2}$, and hence the Goursat problem has no solution either. The same will be obtained if, instead of rational ρ, one takes transcendental ones "sufficiently close" to them (4). If, however, ρ is an algebraic number of arbitrary order N, then the equations

$$(1 + e^{2\pi i m\rho}) f_2^{(m)}(0) = \Phi^{(m)}(0)$$

have solutions for all m, and

$$f_2(t) = \sum_{m=0}^{\infty} \frac{f_2^{(m)}(0)}{m!} t^m$$

is an analytic function. Consequently, the Goursat problem has a solution.

Moscow State University
named after M. V. Lomonosov

Received
13 II 1957

CITED LITERATURE

[1] E. Goursat, Leçons sur l' intégration des équations aux dérivées partielles du second ordre, 2, Note II, 1926.
[2] N. M. Günter, Matem. sborn., 32, 26 (1924).
[3] S. L. Sobolev, Matem. sborn., 38, No. 3, 107 (1932).
[4] A. Ya. Khinchin, Continued Fractions, Moscow, 1949.
[5] J. Meray, J. Riquier, Ann. de l' École Norm., S. III, 7 (1890).

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.