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Abstract

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ON THE THEORY OF NONSTATIONARY DISCONTINUITIES IN RELAXING MEDIA

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In supersonic flows, when a medium passes through surfaces of discontinuity, states of incomplete statistical equilibrium arise, requiring the introduction of additional parameters for their description. The system of hydrodynamic equations in this case must be supplemented by equations describing relaxation processes. One-dimensional steady flows with allowance for relaxation processes were considered in ([1],[2]). Of theoretical interest is the case in which, for a sufficiently small intensity of the shock wave, the velocity of its propagation (relative to the undisturbed medium) proves to be less than the high-frequency sound velocity c_∞, calculated under the assumption that the relaxation processes are frozen. From simple considerations connected with the properties of the Hugoniot adiabat, it follows that in this case a surface of discontinuity does not exist and the state of the medium changes smoothly in a certain transition region.

However, initial conditions may be realized under which a discontinuity of small intensity arises in a relaxing medium. In connection with the indicated impossibility of the existence of a steady picture of the motion, the question arises of the behavior of such a discontinuity and of the process of its degeneration into a region of smooth transition. Below an analytical treatment of this problem is given.

The hydrodynamic equations, supplemented by equations of relaxation processes, in the case of small perturbations have solutions of the type

$$u(x,t) \simeq e^{i(kx-\omega t)}, \tag{1}$$

where

$$k^2 = \omega^2 \frac{1 - i\omega\tau}{c_0^2 - c_\infty^2 i\omega\tau}; \tag{2}$$

u is the velocity of the medium along the x-axis; ω is the frequency; τ is the relaxation time (for simplicity we consider one relaxing parameter); c_0 and c_∞ are the sound velocities, respectively, for limiting small ($\omega\tau \ll 1$) and limiting large ($\omega\tau \gg 1$) frequencies ([3]).

Let us consider a perturbation $u(x,t)$ satisfying the following initial conditions

$$u(x,0) = 0 \qquad \text{for } x \geqslant 0; \tag{3}$$

$$u(0,t) = \begin{cases} 0, & \text{for } t < 0, \\ u_0 = \text{const}, & \text{for } t > 0. \end{cases} \tag{4}$$

These conditions correspond to a perturbation propagating in a medium at rest in a cylindrical tube whose axis coincides with the x-axis, if a piston situated in the plane $x = 0$, at the instant of time $t = 0$, instantaneously changes its velocity from zero to u_0.

Representing the function $u(0,t)$ in the form

$$u(0,t) = -\frac{1}{2\pi i}\int_\Omega \frac{d\omega}{\omega} e^{i\omega t}, \tag{5}$$

where the integration in the plane of the complex variable ω is carried out along the real axis from $-\infty$ to $+\infty$, bypassing the point zero from above; on the basis of (1) we obtain a solution satisfying (4) in the form

$$u(x,t) = -\frac{u_0}{2\pi i}\int_\Omega \frac{d\omega}{\omega} e^{i(kx-\omega t)}. \tag{6}$$

In the positive imaginary half-plane $k(\omega)$ and the integrand in (6) have no singularities. Therefore the path of integration in (6) can be displaced to infinity in the direction of the positive imaginary axis. In this case, asymptotically,

$$\exp\{i(kx-\omega t)\} \simeq$$

$$\simeq \exp\left\{i\omega\left(\frac{x}{c_\infty} - t\right)\right\}$$

and for $x/c_\infty - t > 0$ the integrand in (6) decreases exponentially. Thus $u(x,t) = 0$ for $x/c_\infty - t > 0$. It follows that the solution (6) also satisfies condition (3). Let us consider the disturbance near the line $x - c_\infty t = 0$, regarding the variable $t' = t - x/c_\infty$ as positive and sufficiently small. We deform the path of integration in (6) into a semicircle of large radius R with center at the point O and the adjoining portions of the real axis (Fig. 1).

Fig. 1

The same integral, taken along the path shown by the dotted line, is equal to zero, since when the integration path is taken to infinity (downwards) the

Fig. 1

Figure 1: Fig. 1

integrand decreases exponentially ($t' > 0$). Therefore in (6) the circumference of a circle of radius R with center at the point 0 may be taken as the path of integration. Expanding $k(\omega)$ in a series in $1/\omega\tau$, assuming $|\omega\tau| \gg 1$ and retaining terms of order $1/(\omega\tau)^2$, we obtain:

$$u(x,t) = u_0 e^{-\frac{1}{2}\frac{\alpha}{c_\infty \tau}x} J_0\left(-2i\sqrt{\zeta t'}\right), \tag{7}$$

where $J_0(z)$ is the Bessel function with zero index,

$$\zeta = \frac{1}{2}\frac{\alpha}{c_\infty \tau^2}\left(1 - \frac{3}{4}\alpha\right)x, \qquad \alpha = 1 - \frac{c_0^2}{c_\infty^2} > 0. \tag{8}$$

It can be shown that (7) is valid under the condition $t' \ll \tau$.

Thus, in the medium there propagates, with velocity c_∞, a discontinuity whose intensity decreases with distance x according to the law $e^{-x/l}$, where

$$l = \frac{2c_\infty^2}{c_\infty^2 - c_0^2} c_\infty \tau. \tag{9}$$

In the limiting case $c_\infty^2 - c_0^2 = 0$, as was to be expected, a stationary shock wave propagates from the piston.

We now displace the path of integration in (6) in the following manner (Fig. 2).

The branch points ω_1 and ω_2 of $k(\omega)$ lie on the negative imaginary axis

$$\omega_1 = -\frac{i}{\tau}; \qquad \omega_2 = -\frac{i}{\tau}\frac{c_0^2}{c_\infty^2}. \tag{10}$$

For sufficiently large t', the magnitude $e^{-i\omega t'}$ at the branch points is very small. Therefore, in the integral there remains the residue, equal to unity, at the pole O. Thus, near the front of the shock wave ($e^{-t'/\tau} \ll 1$)

$$u(x,t) = u_0. \tag{11}$$

Let us consider the disturbance at a large distance from the piston. The subintegral expression in (6), for real and nonzero ω, decreases exponentially with increasing x; for sufficiently large x, the integral (6) is determined by the neighborhood of the point O. In the expression

$$\frac{\partial u}{\partial t} = \frac{1}{2\pi} \int_{\Omega} d\omega e^{i(kx-\omega t)} \tag{12}$$

the path of integration may be replaced by the real axis, including the point O. Restricting ourselves, in the expansion of k in powers of ω, to the first two terms, carrying out the integration in (12), and passing to $u(x,t)$, we obtain

[Fig. 2]

Fig. 2

$$u(x,t) = \frac{u_0}{2}\,\Phi\left(\beta\,\frac{c_0 t - x}{\sqrt{x}}\right) + \frac{u_0}{2}, \tag{13}$$

where

$$\beta = \sqrt{\frac{c_0}{(c_\infty^2 - c_0^2)\tau}} \tag{14}$$

and $\Phi(z)$ is the probability integral.

The result (13) is valid under the condition

$$\frac{c_\infty}{\sqrt{c_\infty^2 - c_0^2}}\sqrt{\frac{c_0 \tau}{x}} \ll 1. \tag{15}$$

It follows from this that the region in which the disturbance increases from 0 to u_0 has a width Δ of order

$$\Delta \simeq \frac{1}{\beta}\sqrt{x} \tag{16}$$

and propagates in the medium with velocity c_0; the maximum values of $\partial u/\partial x$ lie on the line $x - c_0 t = 0$.

Thus, the general picture of propagation of the disturbance sent by the piston is as follows: the rupture that forms propagates with velocity c_∞, decaying with distance according to an exponential law and gradually giving way to a smoothly increasing disturbance propagating along the line $x - c_0 t = 0$.

The present analysis has been carried out within the framework of linearized equations for infinitesimally small disturbances. The inclusion of nonlinear terms, as is known, leads in the stationary case to a finite width Δ, which becomes indefinitely large as u_0 vanishes, in agreement with (16).

An analysis from the point of view of the hydrodynamic equations of relaxing media (which will be published later) shows that in a plane supersonic flow there exist stationary rupture solutions with an arbitrarily small jump intensity; the absence of stationary rupture solutions with small rupture intensity is characteristic only of one-dimensional flows.

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