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Abstract

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ASYMPTOTICS OF THE EIGENVALUES AND FUNCTIONS OF A CLASS OF ELLIPTIC SYSTEMS

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Let D be a bounded domain of the n-dimensional Euclidean space D_∞, bounded by a Lyapunov-type surface S.* In what follows we shall consider a differential operator of the form

$$A\left(x, \frac{\partial}{\partial x}\right) \equiv \sum_{i,j=1}^{3} A_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{3} A_i(x) \frac{\partial}{\partial x_i} + A_0(x),$$

whose coefficients are real functional square matrices of order 3, defined for values $x = (x_1, x_2, x_3)$ in some bounded domain Ω ($\overline{D} \subset \Omega$) and sufficiently smooth in this domain.

Let $A\left(x, \frac{\partial}{\partial x}\right)$ be an operator of elliptic type: for every real point $\alpha = (\alpha_1, \alpha_2, \alpha_3) \neq 0$ and arbitrary $x \in \Omega$,

$$\det A_2(x, \alpha) = \det \left\{ \sum_{i,j=1}^{3} A_{ij}(x) \alpha_i \alpha_j \right\} \neq 0.$$

It is also assumed that $A\left(x, \frac{\partial}{\partial x}\right)$ is the variational operator of a positive-definite functional. Under these assumptions, using Carleman' s method ([1]) and supplementing it by the method of estimating the regular parts of the corresponding Green matrices given by Ya. B. Lopatinskii ([2,3]), we obtain asymptotic expressions for the eigenvalues λ_k and eigenfunctions (columns) $u_k(x)$ of the problem

$$A\left(x, \frac{\partial}{\partial x}\right) u_k(x) = -\lambda_k u_k(x) \quad (x \in D); \qquad u_k(x) = 0 \quad (x \in S). \qquad (1)$$

1. We consider the system of differential equations

$$\left[A\left(x, \frac{\partial}{\partial x}\right) - \lambda^2 E \right] u(x) = -\Phi(x) \qquad (2)$$

(E is the identity matrix, $\Phi(x)$ is a sufficiently smooth column) and the corresponding system without parameter

$$A\left(x, \frac{\partial}{\partial x}\right) u(x) = -\Phi(x). \tag{3}$$

* All the results of the present work are valid for $n \geq 2$; for simplicity the notation is given for $n = 3$.

For the systems of equations (2) and (3), fundamental matrices $g(x, y; \lambda)$ and $g(x, y)$ are constructed. The fundamental matrix of system (2) is sought in the form

$$g(x, y; \lambda) = g_0(x - y, y; \lambda) + g_1(x, y; \lambda),$$

where $g_0(x-y, y; \lambda)$ is the fundamental matrix of the operator $A_2\left(x, \frac{\partial}{\partial x}\right) - \lambda^2 E$,

$$g_1(x, y; \lambda) = \int_\Omega g_0(x - z, z; \lambda)\, h(z, y; \lambda)\, dz$$

and the matrix $h(z, y; \lambda)$ is found from a system of Fredholm integral equations, which is solvable, as is proved, for large values of the parameter λ by the first Fredholm theorem.

It is proved that the estimates

$$|g_0(x - y, y; \lambda)| \leq \frac{C_0 e^{-\lambda\varepsilon|x-y|}}{|x - y|}, \qquad |g_1(x, y; \lambda)| \leq C_1 e^{-\lambda\varepsilon|x-y|}(|\ln|x - y|| + 1),$$

hold, where $\varepsilon > 0$ is a fixed sufficiently small number and $C_0, C_1 = \text{const}$.

In a completely analogous way, the fundamental matrix $g(x, y)$ of system (3) is defined, and the estimates

$$|g_0(x - y, y)| \leq \frac{\widetilde{C}_0}{|x - y|}, \qquad |g_1(x, y)| \leq \widetilde{C}_1(|\ln|x - y|| + 1)$$

are valid

$$(\widetilde{C}_0, \widetilde{C}_1 = \text{const}).$$

2. For the systems of equations (2) and (3), boundary-value problems of Dirichlet type are posed. By the method of Ya. B. Lopatinskii ([3]), Green matrices $G(x, y; \lambda)$ and $G(x, y)$ are constructed for these problems, and the solutions of the problems themselves are represented by means of the constructed Green matrices.

The Green matrix $G(x, y; \lambda)$ is sought in the form

$$G(x, y; \lambda) = g(x, y; \lambda) - a(x, y; \lambda),$$

where the matrix $a(x, y; \lambda)$ is found from a system of regular integral equations. In view of what was proved in ([2,3]), we have

$$a(x, y; \lambda) = \int_S H(x, z; \lambda)\, g(z, y; \lambda)\, dz,$$

and the estimates

$$|g(z, y; \lambda)| \leq \frac{Ce^{-\lambda\varepsilon|z-y|}}{|z-y|}, \qquad |H(x, z; \lambda)| \leq \frac{C(\lambda)}{|x-z|^{2-\varkappa}},$$

are valid, where $0 < \varkappa \leq 1$ is the exponent in the Lyapunov condition for the surface S; $C = \text{const}$, and $C(\lambda)$ remains bounded as $\lambda \to \infty$.

In a completely analogous way, the Green matrix $G(x, y)$ is constructed, and it is shown that

$$|G(x, y)| \leq \frac{C^*}{|x-y|}, \qquad |a(x, y)| \leq \frac{C_2}{|x-y|^{1-\varkappa}} \qquad (C^*, C_2 = \text{const}).$$

It is proved that the matrix $G(x, y; \lambda)$ is the resolvent of the kernel $G(x, y)$, and that the kernel $G(x, y)$ itself is symmetric. With the aid of the known expansion of the resolvent of a symmetric kernel in eigenfunctions (columns), we obtain

$$\psi(x, \lambda) = \int_D G(x, z; \lambda) G(z, x)\, dz = \int_D G(x, z) G(z, x; \lambda)\, dz = \sum_{k=1}^{\infty} \frac{u_k(x) u_k'(x)}{\lambda_k(\lambda_k + \lambda^2)},$$

where λ_k are the eigenvalues, $u_k(x)$ is a complete orthonormal system of eigenfunctions (columns) of the kernel $G(x, y)$ (the prime on a functional column denotes transposition).

On the basis of the estimates indicated, we obtain

$$\lim_{\lambda\to\infty} \lambda\psi(x; \lambda) = \lim_{\lambda\to\infty} \lambda \int_D g_0(x-z, x; \lambda)\, g_0(z-x, x)\, dz =$$

$$= \frac{1}{(2\pi)^3} \int_{D_\infty} \{[A_2(x,\alpha)][A_2(x,\alpha)+E]\}^{-1}\, d\alpha.$$

Thus,

$$\lim_{\lambda\to\infty} \sqrt{\lambda} \sum_{k=1}^{\infty} \frac{u_k'(x)u_k(x)}{\lambda_k(\lambda_k+\lambda)} = \frac{1}{(2\pi)^3} \int_{D_\infty} \mathrm{Sp}\{[A_2(x,\alpha)][A_2(x,\alpha)+E]\}^{-1}\, d\alpha. \quad (4)$$

3. Applying to the series (4) a known theorem of Tauberian type (see, for example, ([4]), pp. 703-704 and 706-715), we obtain the following asymptotic expressions for the eigenvalues and eigenfunctions (columns) of problem (1):

$$\sum_{k=1}^{n} u_k'(x)u_k(x) \sim \frac{1}{12\pi^4} \int_{D_\infty} \mathrm{Sp}\{[A_2(x,\alpha)][A_2(x,\alpha)+E]\}^{-1}\, d\alpha \cdot \lambda_n^{3/2}; \quad (5)$$

$$n \sim \frac{1}{12\pi^4} \int_D \left\{ \int_{D_\infty} \mathrm{Sp}\{[A_2(x,\alpha)][A_2(x,\alpha)+E]\}^{-1}\, d\alpha \right\} dx \cdot \lambda_n^{3/2}. \quad (6)$$

It has been proved that formula (6) also holds in the case of the more general operator

$$\widetilde{A}\left(x, \frac{\partial}{\partial x}\right) = A\left(x, \frac{\partial}{\partial x}\right) + \sum_{i=1}^{3} A_i^*(x) \frac{\partial}{\partial x_i} + A_0^*(x)$$

with sufficiently smooth matrices $A_i^*(x)$ and $A_0^*(x)$.

From formulas (5) and (6) there follow, in particular, asymptotic expressions for the eigenvalues and eigenvector-functions of the system of equations of the theory of elasticity (Lamé equations) in the case of the problem with zero displacements of the boundary points, i.e. the corresponding results of Weyl ([5]), formula (70), and Pleijel ([6]), formula (243). From these same formulas there follow, in particular, the results of Carleman ([1]), Theorem VI, and Browder ([7]) in the case of a single elliptic equation of the second order.

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named after Ivan Franko

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Note: Figure translations are in progress. See original paper for figures.