



Soviet-era science, translated into English

V. B. LIDSKII

A differential equation is considered

1957

SovietRxiv

View the original and related papers at <https://sovietrxiv.org/items/ru-195701.71076>

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

Abstract

Full Text

V. B. LIDSKII

**CONDITIONS FOR COMPLETE CONTINUITY OF
THE RESOLVENT OF A NON-SELF-ADJOINT DIF-
FERENTIAL OPERATOR**

(Presented by Academician A. N. Kolmogorov on 27 IX 1956)

A differential equation is considered

$$-y'' + p(x)y = \lambda y \tag{1}$$

with complex function $p(x)$. It is assumed that $p(x)$ has the form $p(x) = q(x) + ir(x)$, where $q(x)$ and $r(x)$ are real-valued functions, summable and bounded on each finite interval of the real axis. Along with equation (1) we consider the associated operator

$$Ly = -y'' + p(x)y, \tag{2}$$

acting in the Hilbert space $L_2(-\infty, +\infty)$. The domain of definition D of the operator L consists of the functions $y(x) \in L_2(-\infty, +\infty)$, absolutely continuous on each finite interval together with their derivative, and such that

$$-y'' + p(x)y \in L_2(-\infty, +\infty).$$

In the present paper conditions imposed on $p(x)$ are given under which the operator L has a completely continuous resolvent and, consequently, a discrete spectrum.

In the case of a real-valued function $p(x)$ bounded from below, a necessary and sufficient condition for the complete continuity of the resolvent of the self-adjoint operator (2) was given by A. M. Molchanov in his dissertation ⁽¹⁾. We also note that, in the nature of the questions considered, the present work is close to the article of M. A. Naimark ⁽²⁾. We now proceed to the statement of the results.

Theorem 1. *Let the function $q(x)$, which is the real part of $p(x)$, satisfy the condition*

$$\lim_{|x| \rightarrow \infty} q(x) = +\infty. \tag{3}$$

Then, whatever the function $r(x)$, which is the imaginary part of $p(x)$, may be, the operator $Ly = -y'' + p(x)y$ has a completely continuous resolvent, and hence a discrete spectrum.

Proof. Choose a real value of the parameter $\lambda = \lambda_0$ for which $q(x) - \lambda_0 \geq 1$ ($-\infty < x < +\infty$). With the aid of constructions close to those carried out by H. Weyl in the paper ⁽³⁾ in the case of a self-adjoint problem, it can be shown that equation (1) for $\lambda = \lambda_0$ has two linearly independent solutions $\psi_1(x)$ and $\psi_2(x)$, belonging respectively to $L_2(0, +\infty)$ and $L_2(-\infty, 0)$. Using these solutions, one can construct a bounded integral operator inverse to $L - \lambda_0 E$. Denote it, as usual, by R_{λ_0} , and show that R_{λ_0} is a completely continuous operator. For this we shall prove that the set of functions $\{y\} : y = R_{\lambda_0} f$, where $\|f\| \leq 1$, is compact in $L_2(-\infty, +\infty)$. By the definition of R_{λ_0} , we have: $-y'' + (q(x) + ir(x))y - \lambda_0 y = f$. Multiplying

this relation by $\bar{y}(x)$, take the integrals of both sides over the limits from $-\infty$ to $+\infty$, and equate the real parts. As a result we arrive at the equality

$$\int_{-\infty}^{+\infty} |y'|^2 dx + \int_{-\infty}^{+\infty} (q(x) - \lambda_0) |y|^2 dx = \operatorname{Re} \int_{-\infty}^{+\infty} f \bar{y} dx. \quad (4)$$

Putting $R_{\lambda_0} f$ in place of y in the right-hand side of this formula and estimating it by the Cauchy-Bunyakovsky inequality, we obtain

$$\int_{-\infty}^{+\infty} (q(x) - \lambda_0) |y|^2 dx \leq \|R_{\lambda_0}\|. \quad (5)$$

This inequality is valid for all functions of the set $\{y\}$. Let now $\varepsilon > 0$ be given; choose N so large that for $|x| \geq N$ the inequality $q(x) - \lambda_0 \geq 2\|R_{\lambda_0}\|\varepsilon^{-1}$ holds. From formula (5) it then follows that

$$\int_{|x| \geq N} |y|^2 dx \leq \frac{\varepsilon}{2}. \quad (6)$$

Let us now note that the functions $\{y\}$, considered on the finite interval $[-N, N]$, form a compact family. This follows, for example, from the lemma proved in [4].* Having chosen on the interval $[-N, N]$ a finite $\varepsilon/2$ -net for the functions $\{y\}$, extend each function of the net by zero to the whole axis. In view of (6) we obtain a finite ε -net for the set $\{y\}$ on the whole axis. Since ε is arbitrary, the set $\{y\}$ is compact. Consequently, the operator R_{λ_0} is completely continuous. From the identity $R_\lambda - R_{\lambda_0} = (\lambda - \lambda_0)R_\lambda R_{\lambda_0}$ it then follows that the resolvent is completely continuous for all λ for which it is defined. The theorem is proved.

Analogously to Theorem 1, with only some technical complications, Theorem 2 is proved.

Theorem 2. Let the function $r(x)$, which is the imaginary part of $p(x)$, satisfy the condition $\lim_{|x| \rightarrow \infty} r(x) = +\infty$ (or $-\infty$).

Then, whatever the real part of $p(x)$ may be, the operator

$$Ly = -y'' + p(x)y$$

has a completely continuous resolvent and, consequently, a discrete spectrum.

Theorem 2 needs explanation. The point is that if no restrictions are imposed on the real part of $p(x)$, then for some $q(x)$ all solutions of equation (1), for all λ , may belong to $L_2(-\infty, +\infty)$.** In this case the operator L is understood to be an operator acting not on the whole manifold D , but on some submanifold of it consisting of functions satisfying certain boundary conditions at infinity (see in this connection [2]).

In conclusion, for a certain broad class of functions $p(x)$ we shall give a necessary and sufficient condition for the complete continuity of the resolvent of operator (2):

* The compactness of the family $\{y\}$ on the interval $[-N, N]$ under the conditions of the present theorem can be established quite simply on the basis of the inequality $\int_{-\infty}^{+\infty} |y'|^2 dx < C$, which easily follows from (4). We deliberately refer to a more general assertion, so that by means of the scheme presented it would be possible to reconstruct the proof of Theorem 2 formulated below.

** We emphasize that under the conditions of Theorem 1 (where $q(x)$ is bounded below) this is impossible.

Theorem 3. Let the function $q(x)$ be bounded below, and the function $r(x)$ be semibounded*. Then, for the complete continuity of the resolvent of the operator

$$Ly = -y'' + (q(x) + ir(x))y$$

it is necessary and sufficient that, for any sequence of nonintersecting intervals $\{D_n\}$ of the same length,

$$\int_{D_n} (q(x) + |r(x)|) dx \rightarrow \infty,$$

when the interval goes to infinity**.

For definiteness we shall assume $r(x)$ to be bounded below. We shall also assume, without loss of generality, that $q(x) \geq 1$ and $r(x) \geq 1$. In this case, as we have already indicated, the operator L has a bounded inverse L^{-1} . We shall first

dwelling on the proof of the sufficiency of the condition of the theorem. Denote by l the set of functions from D that are equal to zero outside a finite interval. It can be proved that, under the condition $q(x) \geq 1$, the set Lu , where $u \in l$, is everywhere dense in $L_2(-\infty, +\infty)$. Bearing in mind that we shall use this remark below, represent Lu in the form

$$Lu = -u'' + (q(x) + r(x))u + (i - 1)r(x)u = Au + (i - 1)ru. \quad (7)$$

By A we have denoted the self-adjoint operator $-d^2/dx^2 + q(x) + r(x)$, which, in view of the positivity of the function $q(x) + r(x)$, is positive definite. Let us note here that, according to a theorem of A. M. Molchanov ((1), p. 175), the operator A has a completely continuous inverse A^{-1} . We next transform formula (7) as follows:

$$Lu = A^{1/2} (E + (i - 1)A^{-1/2}rA^{-1/2}) A^{1/2}u. \quad (8)$$

It is not difficult to check that all operations on the right-hand side of this formula are successively executable. The operator applied to the function u on the right-hand side of formula (8) has a completely continuous inverse. Indeed, the operator $A^{-1/2}rA^{-1/2}$ is a bounded self-adjoint operator***. This fact can be proved analogously to Lemma 1 of our paper (5). Since, moreover, $\text{Im}(i - 1)^{-1} \neq 0$, the operator standing in the parentheses has a bounded inverse. On the other hand, the operator $A^{-1/2}$ is completely continuous together with A^{-1} .

Therefore

$$A^{-1/2} (E + (i - 1)A^{-1/2}rA^{-1/2})^{-1} A^{-1/2}$$

is also completely continuous. From formula (8) it follows that L^{-1} , being bounded, coincides on the everywhere dense set $\{Lu\}$ with a completely continuous operator. Hence L^{-1} is completely continuous. The first part of the theorem is proved.

We now outline in general terms the proof of the necessity of the condition of the theorem. Suppose that L^{-1} is completely continuous, but the condition of the theorem is not satisfied. Then there will be found a sequence of nonintersecting intervals $\{D'_n\}$ of the same length $2d$, for which

$$\int_{D'_n} (q(x) + |r(x)|) dx < C. \quad (9)$$

Let us bring all D'_n into coincidence with the interval $[-d, +d]$ and consider the infinite system of equations $-y'' + p_n(x)y = 0$, $x \in [-d, d]$, $n = 1, 2, \dots$; $p_n(x)$ coincides with $p(x)$ at the corresponding points of D'_n . It can be proved that on

* That is, bounded either below or above.

** Cf. (1), pp. 173-175.

*** Strictly speaking, its closure has this property.

on $[-d, d]$ there exists a function $\nu(x)$, the integral of the square of whose modulus is equal to one and which is orthogonal on $[-d, d]$ to all solutions of all the indicated equations. Consequently, each equation

$$-y'' + p(x)y = \bar{\nu}(x)$$

has on the interval $[-d, d]$ a nonzero solution $\mu_n(x)$, which, together with its derivative, vanishes at the endpoints of the interval.

Consider two families of functions $\{g_n(x)\}$ and $\{f_n(x)\}$ possessing the following properties: each function with index n is nonzero only on D'_n ; the values of $g_n(x)$ on D'_n coincide with the values of $\mu_n(x)$ at the corresponding points of the interval $[-d, d]$, while the values of $f_n(x)$ on D'_n coincide with the values of $\bar{\nu}(x)$ on $[-d, d]$. Obviously, then $\|f_n(x)\| = 1$ and $g_n = L^{-1}f_n$. In view of the complete continuity of the operator L^{-1} , the family $\{g_n\}$ is compact. Hence, owing to the orthogonality of the functions g_n , it follows that $\|g_n\| \rightarrow 0$ as $n \rightarrow \infty$. It can be shown that this fact entails the uniform convergence to zero of the functions $\mu_n(x)$ on the interval $[-d, d]$. Taking this into account, multiply each of the identities

$$\bar{\nu}(x) = -\mu_n'' + (q_n + ir_n)\mu_n$$

by $e^{im\pi x/d}$ and integrate from $-d$ to d . Integrating the first of the integrals on the right twice by parts, we arrive, after simple estimates, at the inequality

$$\left| \int_{-d}^{+d} \bar{\nu}(x) e^{i\frac{m\pi x}{d}} dx \right| \leq \max |\mu_n| \left(\frac{2m^2\pi^2}{d} + \int_{-d}^{+d} q_n dx + \int_{-d}^{+d} |r_n| dx \right).$$

Since, as $n \rightarrow \infty$, the expression in parentheses remains bounded by assumption, while $\max |\mu_n| \rightarrow 0$, it follows from this inequality that all Fourier coefficients of the function $\nu(x)$ are equal to zero. This, however, contradicts the condition

$$\int_{-d}^{+d} |\nu(x)|^2 dx = 1.$$

The contradiction obtained proves the assertion of the theorem.

Moscow Institute of Physics and Technology

Received
25 IX 1956

REFERENCES

- ¹ A. M. Molchanov, *Tr. Mosk. matem. obshch.*, **2** (1953).
- ² M. A. Naimark, DAN, **85**, No. 1 (1952).
- ³ G. Weyl, *Math. Ann.*, **68**, 222 (1910).
- ⁴ V. B. Lidskii, DAN, **112**, No. 6 (1957).
- ⁵ V. B. Lidskii, DAN, **110**, No. 2 (1956).

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.