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Abstract

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MATHEMATICS

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ON THE COEFFICIENTS OF TYPICALLY REAL FUNCTIONS

(Presented by Academician V. I. Smirnov, 1 II 1957)

A function $f(z)$ is called typically real in the disk $|z| < 1$ if it is real for real z, and at the remaining points of this disk satisfies the condition

$$\operatorname{Im}(f(z)) \cdot \operatorname{Im}(z) > 0. \tag{1}$$

Let T be the class of typically real functions $f(z)$, regular in $|z| < 1$ and normalized by the conditions $f(0) = 0$, $f'(0) = 1$. In the present article we consider the subclass $T^{(2)}$ of functions $f_2(z) \in T$ such that also $\tilde{f}_2(z) = \frac{1}{i} f_2(iz) \in T$ ([1]). These functions, in addition to condition (1), satisfy the condition

$$\operatorname{Re}(f_2(z)) \cdot \operatorname{Re}(z) > 0, \tag{2}$$

and all of them are odd. Put

$$f_2(z) = z + x_1 z^3 + x_2 z^5 + \cdots + x_{nz}^{2n+1} + \cdots. \tag{3}$$

We note that the class $T^{(2)}$ contains all odd univalent functions of the form (3) with real coefficients.

For the coefficients x_n the following sharp estimates are known ([1,2]):

$$|x_n| + |x_{n-1}| \leqslant 2 \qquad (n = 2, 3, \ldots); \tag{4}$$

$$-1 \leqslant x_1 \leqslant 1, \qquad -\frac{1}{2} \leqslant x_2 \leqslant \frac{3}{2}. \tag{5}$$

There are no sharp estimates of the individual coefficients x_n for $n > 2$.

Theorem. If $f_2(z) \in T^{(2)}$, then

$$-1 - \frac{\sqrt{3}}{18} \leqslant x_3 \leqslant 1 + \frac{\sqrt{3}}{18} = 1.09\ldots; \tag{6}$$

$$x_{2k} \leqslant \frac{3}{2} \qquad (k = 1, 2, \dots); \tag{7}$$

$$x_4 \geqslant -\frac{2}{3}; \tag{8}$$

$$x_6 \geqslant -\frac{5}{16} - \frac{121}{48\sqrt{33}} \simeq -\frac{3}{4}. \tag{9}$$

The estimates are sharp, and each bound is attained by its own unique function belonging to the class $T^{(2)}$.

Proof. Let $f_2(z) \in T^{(2)}$. Put

$$\varphi(z) = \frac{1 - z^2}{z} f_2(z^{1/2}) = 1 + \alpha_1 z + \cdots + \alpha_{nz}^{n} + \cdots. \tag{10}$$

The relations hold

$$\alpha_n = \left(2x_n + 2x_1 x_{n-1} + \cdots + \begin{matrix} \nearrow x_{n/2}^2 \\ \searrow 2x_{(n-1)/2} x_{(n+1)/2} \end{matrix}\right) - \left(2x_{n-2} + 2x_1 x_{n-3} + \cdots + \begin{matrix} \nearrow x_{n/2-1}^2 \\ \searrow 2x_{(n-3)/2} x_{(n-1)/2} \end{matrix}\right) \begin{matrix} (n \text{ even}) \\ (n \text{ odd}) \end{matrix} \tag{11}$$

$$(n = 1, 2, \dots; \quad x_0 = 1; \quad x_n = 0, \text{ if } x < 0).$$

The function (10) is regular in the disk $|z| < 1$, satisfies there the condition $\mathrm{Re}(\varphi(z)) > 0$ ([1], [2]), and all coefficients α_n are real. Conversely, to every function $\varphi(z)$ possessing these properties there corresponds, by (10), a function $f_2(z) \in T^{(2)}$. We shall denote the class of such functions $\varphi(z)$ by R.

Put

$$\delta_n = \begin{vmatrix} 2 & \alpha_1 & \alpha_2 & \cdots & \alpha_n \\ \alpha_1 & 2 & \alpha_1 & \cdots & \alpha_{n-1} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \alpha_n & \alpha_{n-1} & \alpha_{n-2} & \cdots & 2 \end{vmatrix} \qquad (n = 1, 2, \dots). \tag{12}$$

By Carathéodory's theorem ([3]), the conditions $\delta_n \geqslant 0$ $(n = 1, 2, \dots)$ are necessary and sufficient for $\varphi(z) \in R$; moreover, if $\delta_{n_0} = 0$, then for $n > n_0$ all $\delta_n = 0$. Hence, taking into account the relations (11), one can obtain estimates for x_n in terms of $x_1, \dots, x_{n-1}$. Let, for definiteness, $n = 2k$. Introduce the notation

$$\Delta_1^{(k)} = \Delta(\alpha_1, \dots, \alpha_{2k}) = \begin{vmatrix} 2-\alpha_{2k} & \alpha_1-\alpha_{2k-1} & \cdots & \alpha_{k-1}-\alpha_{k+1} \\ \alpha_1-\alpha_{2k-1} & 2-\alpha_{2k-2} & \cdots & \alpha_{k-2}-\alpha_k \\ \cdot & \cdot & \cdot & \cdot \\ \alpha_{k-1}-\alpha_{k+1} & \alpha_{k-2}-\alpha_k & \cdots & 2-\alpha_2 \end{vmatrix} \qquad (k=1,2,\dots); \tag{13}$$

$$\Delta_2^{(k)} = \Delta_2(\alpha_1, \dots, \alpha_{2k}) = \begin{vmatrix} 1 & \alpha_1 & \cdots & \alpha_k \\ \alpha_1 & 2+\alpha_2 & \cdots & \alpha_{k-1}+\alpha_{k+1} \\ \cdot & \cdot & \cdot & \cdot \\ \alpha_k & \alpha_{k-1}+\alpha_{k+1} & \cdots & 2+\alpha_{2k} \end{vmatrix}. \tag{14}$$

Performing elementary transformations of the determinant (12) ($n = 2k$), we obtain $\delta_{2k} = \Delta_1^{(k)} \cdot \Delta_2^{(k)}$, whence, on the basis of Carathéodory' s conditions,

$$\frac{\begin{vmatrix} 1 & \alpha_1 & \cdots & \alpha_k \\ \alpha_1 & 2+\alpha_2 & \cdots & \alpha_{k-1}+\alpha_{k+1} \\ \cdot & \cdot & \cdot & \cdot \\ \alpha_k & \alpha_{k-1}+\alpha_{k+1} & \cdots & 2 \end{vmatrix}}{\Delta_2(\alpha_1, \dots, \alpha_{2k-2})} \leqslant \alpha_{2k} \leqslant \frac{\begin{vmatrix} 2 & \alpha_1-\alpha_{2k-1} & \cdots & \alpha_{k-1}-\alpha_{k+1} \\ \alpha_1-\alpha_{2k-1} & 2-\alpha_{2k-2} & \cdots & \alpha_{k-2}-\alpha_k \\ \cdot & \cdot & \cdot & \cdot \\ \alpha_{k-1}-\alpha_{k+1} & \alpha_{k-2}-\alpha_k & \cdots & 2-\alpha_2 \end{vmatrix}}{\Delta_1(\alpha_1, \dots, \alpha_{2k-2})} \tag{15}$$

$$(k=1,2,\dots; \quad \Delta_1^{(0)} = \Delta_2^{(0)} = 1).$$

Denote by $A_1(x_1, \dots, x_{2k-1})$ and $A_2(x_1, \dots, x_{2k-1})$ the functions of $2k-1$ variables $x_1, \dots, x_{2k-1}$ which are obtained as a result of the substitution

respectively into the right- and left-hand inequalities (15), instead of $\alpha_1, \dots, \alpha_{2k-1}$, their expressions (11) in terms of $x_1, \dots, x_{2k-1}$; we obtain

$$A_2(x_1, \dots, x_{2k-1}) - B(x_1, \dots, x_{2k-1}) \le 2x_{2k} \le \tag{16}$$

$$\le A_1(x_1, \dots, x_{2k-1}) - B(x_1, \dots, x_{2k-1}),$$

where

$$B(x_1, \dots, x_{2k-1}) =$$

$$= (2x_1x_{2k-1} + 2x_2x_{2k-2} + \dots + x_k^2) - (2x_{2k-2} + 2x_1x_{2k-3} + \dots + x_{k-1}^2).$$

Analogous inequalities can also be written for $2x_{2k+1}$.

We shall assign to each series (3) a point x with Cartesian coordinates $x_1, \dots, x_n$ in the n-dimensional Euclidean space R_n $(n \geq 1)$. To the class $T^{(2)}$ there will correspond in R_n a closed convex body $T_n^{(2)}$, since, if $f_2^{(1)}(z) \in T$ and $f_2^{(2)}(z) \in T^{(2)}$, then also $\lambda f_2^{(1)}(z) + (1-\lambda) f_2^{(2)}(z) \in T^{(2)}$ for any λ, $0 \leq \lambda \leq 1$. Taking the equality sign in relations (16), we obtain the equation of the part $T_{n-1}^{(2)}$ of the boundary of this body that is convex in the direction of the positive, respectively negative, axis x_n.

If within $T_{n-1}^{(2)}$ there exist stationary points of the functions $A_i(x_1, \dots, x_{2k-1}) - B(x_1, \dots, x_{2k-1})$ $(i = 1, 2)$, then their values at these points, in view of the convexity of $T_n^{(2)}$, will respectively be the upper and lower exact bounds of $2x_n$. In this way, for $n = 1, 2$ we find the estimates (5), and for $n = 3$ we obtain the estimates (6), the equality sign on the right in (6) being attained for

$$x_1 = 1 - \frac{1}{\sqrt{3}}, \quad x_2 = 1 - \frac{1}{2\sqrt{3}},$$

and on the left for

$$x_1 = -1 + \frac{1}{\sqrt{3}}, \quad x_2 = 1 - \frac{1}{2\sqrt{3}}.$$

The uniqueness of the extremal functions follows from the fact that at these stationary points there is a strict extremum. Determination of the stationary points and their investigation for arbitrary n is very difficult.

To obtain the estimates (7)–(9), consider the function

$$\psi_2(z) = \frac{f_2(z) + \overline{f_2(\overline{z})}}{2} = z + x_2 z^5 + \dots + x_{2n} z^{4n+1} + \dots \tag{17}$$

It is not hard to see that $\psi_2(z) \in T^{(2)}$. To the class of functions $\psi_2(z)$ defined by formula (17) there will correspond a closed and convex body $\widetilde{T}_n^{(2)}$. Let, for definiteness, $n = 2k$. For the part of the boundary of this body convex in the direction of the positive axis x_{2n}, we have the equation

$$2x_{4k} = A_1(x_2, x_4, \dots, x_{4k-2}) -$$

$$-(2x_2 x_{4k-2} + 2x_4 x_{4k-4} + \dots + x_{2k}^2) + (2x_{4k-2} + 2x_2 x_{4k-4} + \dots + 2x_{2k-2} x_{2k}). \tag{18}$$

It is not hard to notice that the right-hand side of (18) has the stationary point $x_2 = x_4 = \dots = x_{4k-2} = 1$, lying inside $\widetilde{T}_{n-1}^{(2)}$. The corresponding values are

$\alpha_n = 1$ ($n = 1, 2, \dots, 2k-1$), whence we find $x_{4k} = 3/2$. The same is obtained also for $n = 2k+1$. Hence the estimates (7) are obtained.

The uniqueness of the corresponding extremal functions follows from the fact that the right-hand side of (18) has a maximum at this stationary point.

For $n = 2, 3$, by the same method we find exact estimates also from below. Estimate (8) is attained for $x_2 = 1/3$, estimate (9) is attained for

$$x_2 = \frac{\sqrt{33}-3}{6}, \quad x_4 = \frac{9-\sqrt{33}}{24}.$$

The theorem is proved.

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References

1. G. M. Goluzin, *Matem. sborn.*, **27** (69): 2, 201 (1950).
2. W. Rogosinski, *Math. Zs.*, **35**, 93 (1932).
3. C. Carathéodory, *Rend. Circ. Math. Palermo*, **32**, 193 (1911).

Note: Figure translations are in progress. See original paper for figures.