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Abstract

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MATHEMATICS

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SOME THEOREMS ON THE DECOMPOSITION OF INFINITELY DIVISIBLE LAWS

Let X be a random variable with an infinitely divisible (i.d.) law $F(x)$. Its characteristic function (c.f.) $\varphi(t)$ is uniquely represented by the formula

$$\ln\varphi(t) = \beta it - \gamma t^2 + \int_{-\infty}^{0}\left(e^{itx} - 1 - \frac{itx}{1+x^2}\right)dM(x) + \int_{0}^{\infty}\left(e^{itx} - 1 - \frac{itx}{1+x^2}\right)dN(x), \tag{1}$$

where β is a real number; $\gamma \geq 0$; $M(x)$ and $N(x)$ are nondecreasing; $M(-\infty) = N(\infty) = 0$;

$$\int_{-a}^{0} x^2\, dM(x) + \int_{0}^{a} x^2\, dN(x) < \infty, \quad a > 0.$$

If X_1 and X_2 are independent random variables with distribution laws F_1 and F_2, and $X = X_1 + X_2$, $F = F_1 * F_2$ (composition), then we shall speak of a decomposition of the law F into components. We shall be interested in possible decompositions of i.d. laws into components. Denote the class of i.d. laws by I. With respect to the functions $M(x)$ and $N(x)$ we shall say that they characterize the Poisson spectrum of the random variable X; the values $x > 0$ will give the positive, and $x < 0$ the negative, Poisson frequencies. In this note, for the time being, only i.d. laws with positive Poisson spectrum will be considered, i.e. $dM(x) = 0$. If $dN(x) = 0$ for $x \geq X_0 > 0$, then we shall speak of a bounded Poisson spectrum; the case of a finite or countable Poisson spectrum will be given by a c.f. under the condition

$$\ln\varphi(t) = \beta_1 it - \gamma t^2 + \sum_{m=1}^{N}\lambda_m\left(e^{it\mu_m} - 1\right), \tag{2}$$

where $N < \infty$ or $N = \infty$; $\lambda_m > 0$; $\mu_m > 0$; the series $\sum\lambda_m$ converges. The separate Poisson component Y_m has c.f. $\exp\{\lambda_m(e^{it\mu_m} - 1)\}$, and here $\mu_m = D(Y_m)/E(Y_m)$.

We shall be interested in the class $I_0 \subset I$ of i.d. laws that can be decomposed only into i.d. components. It is easy to see that laws from I_0 with a finite or countable Poisson spectrum (c.f. of type (2)) have the same components, i.e. components expressible by a formula of type (2) with the same spectrum (allowing some λ_m to be equal to zero). In simplified terms, the components of laws from I_0 have the same form as these laws themselves. In the work of H. Cramér [1] it was proved that the normal law belongs to I_0 (i.e., it can have only normal components); in the works of D. A. Raikov [2, 3] it was proved that the Poisson law belongs to I_0 (has only Poisson

components); in the author’ s papers ([4,5]) it was proved that the composition of Gaussian and Poisson laws belongs to I_0. The method of ([4,5]) can be carried over to a more general case, and the theorems stated below are obtained.

For laws with a positive finite or denumerable Poisson spectrum we introduce the concept of rationality of the spectrum. A Poisson spectrum will be called **rational** if $\mu_m/\mu_l = r_{lm}$ is rational for any l and m.

If a finite or denumerable Poisson spectrum is not rational, and $\gamma > 0$ (a Gaussian component is present), then the membership of the law in I_0, apparently, is a quite exceptional phenomenon, which will be considered in another paper.

In the present paper we restrict ourselves only to the case of a rational spectrum. For it the following theorems hold.

Theorem 1. *Let F be an infinitely divisible law with a bounded positive rational finite or denumerable Poisson spectrum and with $\gamma > 0$ (having a Gaussian component). In order that $F \in I_0$ (have only infinitely divisible components), it is necessary and sufficient that the Poisson frequencies μ_m in representation (2) coincide with a decreasing sequence of numbers*

$$\mu, \quad \frac{\mu}{k_1}, \quad \frac{\mu}{k_1 k_2}, \quad \frac{\mu}{k_1 k_2 k_3}, \dots, \quad \frac{\mu}{k_1 k_2 \dots k_s}, \dots, \tag{3}$$

where $\mu > 0$; $k_1, k_2, k_3, \dots$ is some set of natural numbers (allowing repetitions).

Thus, if a law $F \in I_0$ has a denumerable spectrum, then it can have only one limiting point of accumulation of frequencies—zero.

Theorem 2. *If in the conditions of Theorem 1 the requirement of boundedness of the spectrum is omitted, then, for F to belong to the class I_0, it is necessary that the Poisson frequencies μ_m in representation (2) coincide with a sequence of numbers*

$$\dots, \quad k_{-3}k_{-2}k_{-1}\mu, \quad k_{-2}k_{-1}\mu, \quad k_{-1}\mu, \quad \mu, \quad \frac{\mu}{k_1}, \quad \frac{\mu}{k_1 k_2}, \quad \frac{\mu}{k_1 k_2 k_3}, \dots, \tag{4}$$

where $\dots, k_{-3}, k_{-2}, k_{-1}, k_1, k_2, k_3, \dots$ is some set of natural numbers.

Whether this condition is sufficient has not yet been clarified.

Theorems 1 and 2 are connected with Theorem 3.

Theorem 3. *Let F be an infinitely divisible law with a semipositive bounded Poisson spectrum, so that $dN(x) = 0$ for $x > a$ in formula (2). Then all its components have characteristic functions of the form*

$$\varphi_1(t) = \exp\left(P_3(it) + t^4 \int_0^a e^{itu}\phi(u)\,du\right), \tag{5}$$

where $P_3(it)$ is a polynomial of degree not higher than the third; $\phi(u)$ is a real function summable with its square on the segment $[0, a]$.

Theorem 4. *Let F be an infinitely divisible law with a semipositive bounded Poisson spectrum which is rational to the right of the point b, so that the characteristic function has the form*

$$\varphi(t) = \exp\left(\beta it - \gamma t^2 + \int_0^b (e^{itx} - 1)\,dN(x) + \sum_{j=1}^{m} \lambda_j \left(\exp\left(it\frac{a_j}{q}\mu\right) - 1\right)\right), \tag{6}$$

where $\lambda_j > 0$; a_j, q are integers; $a_1 < a_2 < \cdots < a_m$; $\frac{a_1}{q}\mu < b$. Let $\gamma > 0$. Then all its components have characteristic functions of the form

$$\varphi_1(t) = \exp\left(P_3(it) + t^4 \int_0^b e^{itu}\phi(u)\,du + \sum_{n=1}^{q} (\alpha_n + \beta_n it) \exp\left(\left(it\frac{n}{q}\mu\right) - 1\right)\right), \tag{7}$$

where α_n, β_n are real numbers, not necessarily positive; $\phi(u)$ is a function summable with its square on $[0, b]$; $P_3(it)$ is a cubic polynomial.

We note that from Theorem 4 it is easy to derive the main result of papers (4,5): a composition of Gaussian and Poisson laws belongs to I_0, i.e., can be decomposed only into compositions of the same kind. For this it suffices in (6) to take $b = 0$; $q = m = a_j = 1$. One obtains the equality

$$\varphi_1(t) = \exp(P_3(it) + a_1(\exp(i\mu t) - 1)).$$

It is readily established that $\alpha_1 > 0$; $P_3(it) = \beta it - \gamma t^2$; $\gamma \geqslant 0$. Theorems 1 and 2 follow from Theorem 4 with the aid of a lemma that is also of independent interest.

Lemma. *Let $\mu > 0$, $\gamma > 0$; $0 < m < M$; m and M are integers, and m does not divide M. Then, for every sufficiently small $\nu > 0$, the function*

$$\psi(t) = \exp\left(-\gamma t^2 + \lambda_1(e^{M\mu it} - 1) + \lambda_2(e^{m\mu it} - 1) - \nu(e^{\mu dit} - 1)\right), \qquad (8)$$

where d is the greatest common divisor of the numbers m and M; $\lambda_1, \lambda_2 > 0$ are given numbers, will be the characteristic function of some random variable.

This lemma is proved by means of the saddle-point method and the transformation formula for ϑ-functions; in this way one also obtains an asymptotic expression for the probability density corresponding to the characteristic function (7).

The main tools for proving Theorems 3 and 4 are the same as in (^{4,5}): the Paley–Wiener theorem on the representation of entire functions of exponential type belonging to L_2 on some axis, and the use of special functions—the "little cups of I. M. Vinogradov."

The indicated tools, in combination with the saddle-point method, make it possible to study, with respect to possible membership in the class I_0, laws with a bounded spectrum of all the remaining types: a continuous spectrum, a finite or countable spectrum that is not rational, and a spectrum containing negative frequencies. This, however, requires a considerable technical complication of the available apparatus and will be done subsequently. The case of an unbounded spectrum so far only partially admits the indicated treatment.

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Note: Figure translations are in progress. See original paper for figures.

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