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Abstract

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MATHEMATICS

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ON THE BOUNDEDNESS IN L_p OF DERIVATIVES OF SOLUTIONS OF ELLIPTIC EQUATIONS AND SYSTEMS

(Presented by Academician V. I. Smirnov on 19 IV 1957)

Consider in n -dimensional space $x(x_1, \dots, x_n)$ a domain Ω , whose boundary S is a closed simply connected surface. Suppose that S , in a neighborhood of each of its points, can be represented in local coordinates by means of a sufficiently many times differentiable function. Let an elliptic system of equations be given in Ω (see, for example, (1))

$$Lu = \sum_{k_1, \dots, k_{2m}=1}^n a^{(k_1, \dots, k_{2m})}(x) \frac{\partial^{2m} u}{\partial x_{k_1} \dots \partial x_{k_{2m}}} + Tu = f(x), \quad (1)$$

where $u(x) = [u_1(x), \dots, u_N(x)]$; $f(x) = [f_1(x), \dots, f_N(x)]$; $a^{(k)}(x)$ are square matrices of order N^2 ; Tu is a linear differential operator of order lower than $2m$.

By a **generalized solution** of system (1) we shall mean such a function* $u \in W_p^{(2m)}(\Omega)$ (2), which almost everywhere in Ω satisfies system (1). We shall consider a generalized solution satisfying the boundary conditions

$$u|_S = \frac{\partial u}{\partial \nu} \Big|_S = \dots = \frac{\partial^{m-1} u}{\partial \nu^{m-1}} \Big|_S = 0, \quad (2)$$

where ν is the direction of the exterior normal to S .

We formulate the main result of the present note.

Theorem. *If the coefficients of system (1) are continuous, $f \in L_p(\Omega)$ ($p > 1$), and there exists a generalized solution of problem (1)–(2), then this solution satisfies the inequality*

$$\|u\|_{W_p^{(2m)}(\Omega)} \leq C_1 \|f\|_{L_p(\Omega)} + C_2 \|u\|_{L_p(\Omega)}, \quad (3)$$

where C_1, C_2 do not depend on u, f .

Inequality (3) for any interior subdomain Ω^* of the domain Ω was obtained in our note ⁽³⁾. For the case $m = 1, N = 1$, inequality (3) was obtained earlier in our notes ^(4,5). The proof of the inequality

* A vector function u belongs to $W_p^{(2m)}(\Omega)$ if each of its N components belongs to $W_p^{(2m)}(\Omega)$. The norm of a vector function is defined by the rule

$$\|u\|_{W_p^{(2m)}} = \left(\sum_{i=1}^N \|u_i\|_{W_p^{(2m)}}^p \right)^{1/p}.$$

(3) in the interior subdomain Ω^* is carried out in the same way as in (3). To obtain inequality (3) in the boundary strip, the following lemma is first proved.

Lemma. Let the domain Ω be the cube $D : 0 \leq x_i \leq 1$ ($i = 1, \dots, n$), and let the coefficients of system (1) be constant. Suppose that in D there exists a generalized solution of system (1) with constant coefficients, satisfying on the boundary $x_n = 0$ the conditions (2). Then, if $f \in L_p(\Omega)$ ($p > 1$), the estimate holds

$$\|u\|_{W_p^{(2m)}(D_r)} \leq A \|f\|_{L_p(D)} + \frac{B}{r^\gamma} \|u\|_{L_p(D)}, \quad (4)$$

where D_r is any subdomain of the cube D whose distance from all faces of D , except $x_n = 0$, is not less than $r > 0$; A, B are positive constants depending only on m, n, p .

For the proof of the lemma, the Green's function of system (1) with constant coefficients is constructed for the half-space $x_n > 0$. In constructing the regular part of the Green's function, a method analogous to that of O. V. Guseva is used, by means of which inequality (3) was proved for $p = 2$ ⁽⁶⁾. After certain transformations, the solution $u(x)$ and its derivatives up to order $2m$ inclusive are represented in the form of Fourier integrals. Under the sign of these integrals stands the Fourier transform of the function f with multipliers satisfying the conditions formulated in a note of S. G. Mikhlin ⁽⁷⁾. The considerations listed lead to inequality (4). To prove estimate (3) in the boundary strip, a part of the boundary S is straightened by means of a change of coordinates. Without loss of generality one may assume that the equation of this part of the boundary will be $x_n = 0$. A system of cubes D_ρ is constructed, one of whose faces coincides with the plane $x_n = 0$ and whose edges have length $\rho > 0$. Inside each

such cube D_ρ , system (1) is replaced by a system with constant coefficients obtained by substituting into the coefficients the coordinates of the center of the corresponding cube. Since the constant A in the corresponding estimate (4) for the cube D_ρ will not depend on ρ , it is possible to apply a method analogous to Schauder's method (see, for example, (4)), which after simple transformations gives estimate (3). We note that if the estimate

$$\|u\|_{L_p(\Omega)} \leq C_3 \|f\|_{L_p(\Omega)}, \quad (5)$$

is known from the very beginning, then estimate (3) will have the form

$$\|u\|_{W_p^{(2m)}(\Omega)} \leq C \|f\|_{L_p(\Omega)}. \quad (6)$$

In some cases estimate (5), and consequently also (6), is easy to obtain if one assumes that system (1) is strongly elliptic (8), more precisely, that on functions $u \in W_p^{(2m)}(\Omega)$ satisfying the equalities (2), the inequality

$$|(Lu, u)| > \mu^2 \|u\|_{W_2^{(m)}(\Omega)}^2, \quad (7)$$

holds, where

$$(u, v) = \sum_{i=1}^N \int_{\Omega} u_i v_i d\Omega$$

and μ is some real constant different from zero. Thus, for example, if inequality (7) is fulfilled, then estimate (6) is obtained for $p > 1$, if $m \leq 2n$, and for $p > \frac{2n}{n+2m}$, if $m > 2n$.

As is known (see, for example, (9)), the a priori estimate (6) makes it possible to prove the existence theorem for a generalized solution by the method of continuation with respect to a parameter.

Let us now consider a quasilinear system of the form

$$\begin{aligned} Mu &= \sum_{\lambda} a_{\lambda}^{(k_1, \dots, k_{2m})}(x; u, Du, \dots, D^{2m-1}u) \frac{\partial^{2m} u}{\partial x_{k_1} \dots \partial x_{k_{2m}}} = \\ &= f_{\lambda}(x; u, \dots, D^{2m-1}u), \end{aligned} \quad (8)$$

where

$$a_{\lambda}^{(k)}(x; u, \dots, D^{2m-1}u) = a^{(k)}(x) + \lambda b^{(k)}(x; u, Du, \dots, D^{2m-1}u),$$

$$f_\lambda(x; u, Lu, \dots, D^{2m-1}u) = f(x) + \lambda\varphi(x, u, Lu, \dots, D^{2m-1}u);$$

by $L^k u$ are denoted the derivatives of the function of order k , and by λ a real parameter.

If the coefficients $a^{(k)}$, $b^{(k)}$, and φ of system (8) are continuous in all variables and $f \in L_p(\Omega)$ for $p > n$, then Mu is an operator mapping $W_p^{(2m)}(\Omega)$ into $L_p(\Omega)$. This follows from the fact that, according to the embedding theorems ⁽²⁾, all derivatives of the function u up to order $2m - 1$, inclusive, will be continuous in Ω .

If, for system (8), when $\lambda = 0$, condition (7) is satisfied with regard to the boundary conditions (2), then it is not difficult, with the aid of the contraction mapping principle, to prove the following theorem.

Theorem. *If the coefficients $a^{(k)}$, $b^{(k)}$, φ are continuous in all variables, the coefficients $b^{(k)}$, φ are differentiable in all variables except x , and $f \in L_p(\Omega)$ for $p > n$, then system (8) with boundary conditions (2) has a generalized solution for $|\lambda| < \varepsilon$, where ε is some positive number.*

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