MATHEMATICS

1957

SovietRxiv

Abstract

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MATHEMATICS

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SURFACES WITH PLANE GENERATORS ALONG WHICH THE TANGENT PLANE IS CONSTANT

(Presented by Academician P. S. Aleksandrov, 5 III 1957)

1. Developable surfaces of three-dimensional space can be generalized to N-dimensional space in two ways. The properties of being developable onto a plane, or of being the envelope of a one-parameter family of planes, also hold in N-dimensional space for n-dimensional surfaces with $(n-1)$-dimensional plane generators along which the tangent plane is constant.

By virtue of the property of developable surfaces of three-dimensional space of consisting of rectilinear generators along which the tangent plane is constant, they are generalized to a broader class of n-dimensional surfaces with p-dimensional $(n-1 \geq p \geq 1)$ plane generators, along which the tangent plane is constant, in N-dimensional space. Surfaces of n dimensions with p-dimensional plane generators along which the tangent plane is constant are of great interest also in connection with the problem of bending multidimensional surfaces ([1,2]).

Conical n-dimensional surfaces formed by p-dimensional planes joining a fixed $(p-1)$-dimensional plane vertex with the points of an $(n-p)$-dimensional directrix surface of general type, and conical n-dimensional surfaces formed by q-dimensional planes joining a fixed $(q-1)$-dimensional plane vertex with the points of an $(n-q)$-dimensional surface with $(p-q)$-dimensional plane generators $(q = p-1, \dots, 1)$, along which the tangent $(n-q)$-dimensional plane is constant, constitute the most numerous class of n-dimensional surfaces with p-dimensional plane generators along which the tangent plane is constant, in N-dimensional space. But only those distinct from conical surfaces with plane generators along which the tangent plane is constant are of great interest. However, of these surfaces only the developable $(p = n-1)$ n-dimensional surfaces ([3]) and bendable hypersurfaces of rank 2 in Euclidean N-dimensional space E_N ([1]) have been well studied.

2. The property of a surface of having plane generators along which the tangent plane is constant is projective in character, and the study of such surfaces is naturally carried out in projective space P_N. To each point of the surface we attach a frame consisting of $N+1$ points $M_0, M_1, \dots, M_N$, not lying in one hyperplane. The point M_0 is the current point of the surface.

The points $M_1, M_2, \dots, M_n$ of the frame are placed in the tangent plane to the surface at the point M_0, with p of them, $M_{n-p+1}, \dots, M_n$, in the plane generator passing through the point M_0. The points $M_{n+1}, \dots, M_N$ are placed outside the tangent plane at the point M_0. The infinitesimal displacement of the frame is written in the form

$$dM_J = \omega_J^K M_K \qquad (J, K = 0, 1, \dots, N).$$

The surface is determined by the Pfaff system of equations

$$\omega_i^{\hat{\alpha}} = \omega_i^{\hat{\alpha}} = 0 \quad (\beta, \gamma = 1, 2, \dots, n-p;\ i = n-p+1, \dots, n;$$

$$\omega_i^{\beta} = a_{i\gamma}^{\beta} \omega^{\gamma}, \quad \omega_{\beta}^{\hat{\alpha}} = a_{\beta\gamma}^{\hat{\alpha}} \omega^{\gamma}; \quad \hat{\alpha} = n+1, \dots, N),$$

where the coefficients are subject to the relations

$$a_{\beta\gamma}^{\hat{\alpha}} = a_{\gamma\beta}^{\hat{\alpha}}, \quad a_{i\beta}^{\nu} a_{\nu\gamma}^{\hat{\alpha}} - a_{i\gamma}^{\nu} a_{\beta\nu}^{\hat{\alpha}} = 0 \quad (\beta, \gamma, \nu = 1, 2, \dots, n-p).$$

The dimension of the osculating planes at the points of an n-dimensional surface with p-dimensional plane generators, along which the tangent plane is constant, is equal to $n + R$, where R is the number of independent vectors

$$a_{\beta\gamma} = \{a_{\beta\gamma}^{n+1}, a_{\beta\gamma}^{n+2}, \dots, a_{\beta\gamma}^{N}\} \quad (\beta, \gamma = 1, 2, \dots, n-p)$$

in the parameter space of $N - n$ dimensions. The number $n + R$ may, for different surfaces, take values from $n + 1$ to

$$n + \frac{1}{2}(n-p)(n-p+1).$$

The dimension $n+R$ of the osculating planes and the number $n-p$ of parameters on which the tangent plane depends are the most essential numerical parameters of the surface. Prescribing these parameters a priori, we can single out narrow classes of surfaces whose structure is amenable to investigation. Thus, for $n + R > n + (n-p)$ we have the following theorem.

Theorem 1. *If the osculating planes at the points of an n-dimensional surface with p-dimensional $(n - p > 1)$ plane generators, along which the tangent plane is constant, have dimension greater than $n+(n-p)$, then the surface is a conical one with a $(p-1)$-dimensional plane vertex.*

Consequently, nonconical n-dimensional surfaces with p-dimensional plane generators, along which the tangent plane is constant, have osculating planes of dimension not exceeding the number $n + (n - p)$.

3. In the case when the osculating planes at the points of the surface have the least dimension $n+1$, the surface, for $n-p>1$, is a hypersurface of the $(n+1)$-dimensional projective space P_{n+1}. The characteristic property of hypersurfaces with plane generators, along which the tangent hyperplane is constant, is given by the following theorem.

Theorem 2. *A hypersurface in P_{n+1} with p-dimensional plane generators, along which the tangent hyperplane is constant, is the envelope of a family of hyperplanes corresponding, under a correlative transformation, to the points of an $(n-p)$-dimensional surface in P_{n+1}.*

The class of hypersurfaces with p-dimensional plane generators, along which the tangent hyperplane is constant, is dual to the class of $(n-p)$-dimensional surfaces without plane generators with a constant tangent plane along them in a projective space of $n+1$ dimensions.

4. For n-dimensional surfaces with $(n-2)$-dimensional plane generators, along which the tangent plane is constant, the dimension of the osculating planes can have the values $n+1$, $n+2$, and $n+3$. The value $n+3$ satisfies the conditions of Theorem 1; the value $n+1$ characterizes the hypersurfaces already considered. For n-dimensional surfaces with $(n-2)$-dimensional generators whose osculating planes have dimension $n+2$, we have Theorem 3.

Theorem 3. *All nonconical n-dimensional surfaces with $(n-2)$-dimensional plane generators, along which the tangent plane is constant, and which have osculating planes of dimension $n+2$, are divided into the following three types:*

a) *one-parameter families of $(n-1)$-dimensional conical surfaces with $(n-2t_0-2)$-dimensional plane vertices and $2t_0$-dimensional*

directing surfaces, which are either developable surfaces or conical $2t_0$-dimensional surfaces with $(2t_0-2)$-dimensional vertices, $t_0 = 1, 2, \dots, [\frac{1}{2}(n-1)]$;

b) the envelopes of the osculating planes of order $\frac{1}{2}n$ to a two-dimensional surface having a conjugate net, or to a two-dimensional surface having one family of asymptotic lines, in the case of even n;

c) the envelopes of the osculating planes of order $\frac{1}{2}(n-1)$ to a three-dimensional manifold of points of a two-parameter congruence of lines, or to a three-dimensional manifold of points of the totality tangent to the asymptotic lines of a two-dimensional surface with one family of asymptotic lines, in the case of odd n.

Thus, for $n-p=2$ we have a complete classification of n-dimensional surfaces with $(n-2)$-dimensional plane generators along which the tangent plane is constant.

In conclusion I express my gratitude to S. P. Finikov, under whose supervision this work was carried out.

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Received
5 III 1957

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Note: Figure translations are in progress. See original paper for figures.