Mathematics

1957

SovietRxiv

Abstract

Full Text

Mathematics

V. G. Vinokurov

CONDITIONS FOR THE REGULARITY OF PROBABILITY PROCESSES

(Presented by Academician A. N. Kolmogorov, 1 XI 1956)

Let an abstract set of states $\Omega = \{\omega\}$ be given, and let Φ be a given set of functions $\varphi(t)$ with values in Ω, where t ranges over the set of all integers. We shall assume that for any φ in Φ and any n, the function $\varphi_n(t) = \varphi(t+n)$ is also contained in Φ. If a Borel field of probabilities P is given on Φ, then this defines a probability process $\{\Omega, \Phi, P\}$ ([1]). For each t, define the mapping R_t of the space Φ into Ω:

$$R_t\varphi = \omega, \qquad \text{if } \varphi(t) = \omega.$$

Denote by ζ_t the partition of the space Φ into disjoint sets on which the mapping R_t is constant, and by O_t the Borel field of measurable subsets of Φ that are sums of elements of the partition ζ_t. We shall assume that every measurable set either belongs to the smallest Borel field O containing all the fields O_t, or differs from some set in O by a null set.

Define on Φ the transformation

$$T\varphi(t) = \varphi(t+1).$$

We shall assume that for any measurable set A, the set TA is also measurable. If $P(TA) = P(A)$, then the process is called stationary.

Let $\mathcal{H}$ be the Hilbert space of square-integrable functions defined on Φ. Then the transformation T corresponds to a unitary operator U defined on $\mathcal{H}$,

$$Ux(\varphi) = x'(\varphi), \qquad \text{where } x'(\varphi) = x(T\varphi).$$

Denote by $O^-(t)$ the smallest Borel field containing all fields $O_s,\ s \le t$, and by $O^+(t)$ the smallest Borel field containing all fields $O_s,\ s > t$.

For a measurable set A, denote by $P(A \mid \varphi^-_{(t)})$ the conditional probability that $\varphi \in A$, given the values $\varphi(s),\ s \le t$. $P(A \mid \varphi^-_{(t)})$ is defined by the equation

$$P(A \cap B) = \int_B P(A \mid \varphi^-_{(t)})\, dP,$$

where B ranges over all sets in the field $O^-(t)$.

Similarly, by $P(A \mid \varphi_t)$ we denote the conditional probability that $\varphi \in A$, given the value $\varphi(t)$.

$P(A \mid \varphi_t)$ is determined from the equation

$$P(A \cap B) = \int_B P(A \mid \varphi_t)\, dP,$$

where B ranges over all sets in O^*_t.

A stochastic process is called Markov if, for all t and all $A \in O^+_{(t)}$,

$$P(A \mid \varphi^-_{(t)}) = P(A \mid \varphi_t)$$

almost everywhere on Φ.

Denote by $\mathcal{H}_t$, $\mathcal{H}^-_{(t)}$, and $\mathcal{H}^+_{(t)}$ the subspaces of $\mathcal{H}$ consisting, respectively, of functions measurable with respect to the fields O_t, $O^-_{(t)}$, and $O^+_{(t)}$.

The Markov condition may now be expressed as follows: for the process to be Markov, it is necessary and sufficient that, for every t, the projection of $\mathcal{H}^+_{(t)}$ onto $\mathcal{H}^-_{(t)}$ coincide with the projection of $\mathcal{H}^+_{(t)}$ onto $\mathcal{H}_t$.

Let $S = \bigcap_t \mathcal{H}^-_{(t)}$; S always contains the constant function. We shall call a process regular if S contains only one constant function. The meaning of regularity is that the course of the process in the future depends arbitrarily little on its behavior in the sufficiently distant past.

1. Lemma. Let $\{\mathcal{H}_n\}$, $n = 1, \dots, \infty$, be a sequence of subspaces of the space $\mathcal{H}$ such that, for $n < m$, $\mathcal{H}_n \supset \mathcal{H}_m$. Denote

$$S = \bigcap_{n=1}^{\infty} \mathcal{H}_n,$$

let r_x be the projection of the element x onto S, and let r^n_x be the projection of x onto $\mathcal{H}_n$.

Then, for any x in $\mathcal{H}$:

1) $\|r_x - r^n_x\| \to 0$ as $n \to \infty$;

2) for any $\varepsilon > 0$ there exists an n such that

$$|(x, y) - (r_x, y)| < \varepsilon \|y\|$$

for all $y \in \mathcal{H}_n$.

With the aid of this lemma the following theorems are proved.

Theorem 1. In order that a stochastic process be regular, it is necessary and sufficient that, for every measurable set A and every $\varepsilon > 0$, there exist a t such that

$$|P(A \cap B) - P(A)P(B)| < \varepsilon$$

for all $B \in O^-_{(t)}$.

Theorem 2. In order that a Markov process be regular, it is necessary and sufficient that, for any t, $A \in O_t$, and $\varepsilon > 0$, there exist an n_0 such that, for $n \geq n_0$,

$$|P(A \cap B) - P(A)P(B)| < \varepsilon$$

for all $B \in O_{t-n}$.

It follows from this theorem that, in the case of a stationary Markov process with a countable number of states, the definition of regularity adopted in this note coincides with the known definition of regularity of homogeneous Markov processes, expressed by the fact that $P^n_{ij} \to P_j$ as $n \to \infty$. Here the transition probabilities are $P^n_{ij} = P^n_{ij}/P_i$, where $P^n_{ij} = P(R^{-1}_t \omega_j \cap R^{-1}_{t-n} \omega_i)$; $P_i = P(R^{-1}_t \omega_i)$. Markov' s theorem, asserting

* Concerning these definitions, see ([2], [3]).

that from the condition $P^n_{ij} > 0$, starting with some n, regularity follows, also follows from this theorem and from the first assertion of the lemma.

For stationary regular processes the unitary operator U has an absolutely continuous spectrum, i.e., for every element x the spectral function $(E_t x, x)$, where E_t is the resolution of the identity for the operator U, is absolutely continuous.

In the case of a Markov process with a countable number of states, using Theorem 2, it is easy to show that, conversely, from the absolute continuity of the spectrum of the operator U follows the regularity of the process.

It can also be shown that for a stationary Markov process with a countable number of states, regularity is equivalent to the fact that the transformation T is mixing in the narrow sense.

The results listed can be carried over also to processes with continuous time.

2. Denote by $O_{t_1 t_2 \dots t_n}$ the smallest Borel field containing the fields $O_{t_1}, O_{t_2}, \dots, O_{t_n}$, by $\mathcal{H}_{t_1 t_2 \dots t_n}$ the subspace consisting of functions measurable with respect to the field $O_{t_1 t_2 \dots t_n}$, and by $\mathcal{H}^l_{t_1 t_2 \dots t_n}$ the smallest subspace containing the subspaces $\mathcal{H}_{t_1}, \mathcal{H}_{t_2}, \dots, \mathcal{H}_{t_n}$.

We shall call a stochastic process linear if, for any $t_1 < t_2 < \dots < t_n$, the projection of the subspace $\mathcal{H}^+_{(t_n)}$ onto $\mathcal{H}_{t_1 t_2 \dots t_n}$ is contained in $\mathcal{H}^l_{t_1 t_2 \dots t_n}$. Denote by $\mathcal{H}^l_{(t)}$ the smallest subspace containing all subspaces $\mathcal{H}_s$, $s \leq t$.

Let $S_l = \bigcap_t \mathcal{H}^l_{(t)}$. For linear processes $S = S_l$. A Markov process may serve as an example of a linear process. In the case of a stationary process with two states ω_1 and ω_2 it can be proved that, under the condition that all numbers

$$P\left(R_1^{-1}\omega_{i_1} \cap R_2^{-1}\omega_{i_2} \cap R_3^{-1}\omega_{i_3}\right) \neq 0,$$

where i_1, i_2, i_3 take the values either 1 or 2, there exist no other linear processes except Markov ones.

The problems considered in this note were suggested to me by A. N. Kolmogorov, to whom I express my gratitude.

Institute of Mathematics and Mechanics
named after V. I. Romanovskii
Academy of Sciences of the Uzbek SSR

Received
1 XI 1956

REFERENCES

1. B. V. Gnedenko, A. N. Kolmogorov, *Collection: Mathematics in the USSR over 30 Years*, 1947.

2. A. N. Kolmogorov, *Basic Concepts of Probability Theory*, Moscow–Leningrad, 1936.

3. J. L. Doob, *Stochastic Processes*, 1953.

Note: Figure translations are in progress. See original paper for figures.