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Abstract

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FERMI FIELDS AND SPINORS OF INFINITE-DIMENSIONAL SPACE

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The development of functional methods in quantum field theory has made it possible to reduce the problem of determining the propagation function of a system of interacting particles to the problem of determining the propagation function of a fermion moving in an arbitrary external field of the corresponding Bose particles, and to subsequent functional integration over this external field ($^{1-3}$). However, the practical realization of this program encounters extremely serious mathematical difficulties, and so far no substantial success has been achieved along this path. In the present note a new aspect of this problem is considered, revealing a somewhat unexpected connection between the theory of Fermi fields and the theory of spinors of infinite-dimensional space.

For definiteness we shall speak of quantum electrodynamics, although all the results carry over directly to any version of meson theory.

Let us write the operator of the quantized electron-positron field $\psi(x)$ in the interaction representation in the form

$$\psi(x) = \sum u_n(x) a_n, \tag{1}$$

where $u_n(x)$ is a complete orthonormal system of solutions of the Dirac equation, and a_n, depending on the sign of the energy, are either electron annihilation operators or positron creation operators, satisfying the usual anticommutation relations

$$[a_m, a_n]_+ = \delta_{mn}. \tag{2}$$

Introduce the quantities $\Gamma_{n\alpha}$ ($\alpha = 1, 2$):

$$\Gamma_{n1} = a_n + a_n^+, \qquad \Gamma_{n2} = \frac{1}{i}(a_n - a_n^+). \tag{3}$$

With the aid of (2) it is easy to verify the validity of the anticommutation relations

$$\Gamma_{m\alpha}\Gamma_{n\beta} + \Gamma_{n\beta}\Gamma_{m\alpha} = 2\delta_{mn}\delta_{\alpha\beta}. \tag{4}$$

In what follows, to shorten the notation, we shall sometimes denote composite indices of the type $(n\alpha)$ by a single letter: A, B, etc.

Relations (4) make it possible, with the aid of the quantities Γ_A, to construct a spinor representation of the rotation group of infinite-dimensional Euclidean space, E_∞, i.e. the group of linear transformations of the variables x_A leaving invariant the quadratic form $\sum x_A^2$. The operators of infinitesimal rotations (“moments”) in this representation have the form

$$M_{AB} = \frac{1}{4i}(\Gamma_A\Gamma_B - \Gamma_B\Gamma_A), \tag{5}$$

and the spinors are the state vectors of the field Ψ. This definition of spinors is a direct generalization to the infinite-dimensional case of Cartan’ s definition of spinors of Euclidean space of n dimensions $^{(4)}$. The connection is especially clear in the occupation-number representation.

With the aid of the operators (5), an arbitrary rotation S can be represented in the form

$$S = \exp\left\{i \sum S_{AB} M_{AB}\right\}, \tag{6}$$

where S_{AB} is an arbitrary antisymmetric real matrix.

Let us consider the connection between the spinor and vector representations of rotations of the space E_∞. To determine the transformation of a vector corresponding to the rotation S, note that the quantity Γ_A must transform as a vector of the space E_∞, i.e., the relations

$$S^{-1}\Gamma_A S = \sum V_{AB}\Gamma_B, \tag{7}$$

must hold, where V_{AB} is the matrix of an orthogonal transformation of the space E_∞. Conversely, if a transformation V is given, then the corresponding transformation S has the form

$$S = \exp\left\{\frac{i}{2}\sum (\ln V)_{AB} M_{AB}\right\}. \tag{8}$$

We shall show that the motion of an electron in an external electromagnetic field $A_\mu(x)$ is described by a rotation of the space E_∞ of the type (6). The general solution of the Schrödinger equation can be constructed if the operator $S(t, t_0)$ is known, satisfying the equation

$$i\frac{\partial S(t, t_0)}{\partial t} = H(t) S(t, t_0) \tag{9}$$

and the initial condition

$$S(t_0, t_0) = I.$$

In our case the Hamiltonian $H(t)$ has the form

$$H(t) = \frac{e}{2} \int [\bar{\psi}(x), \gamma_\mu \psi(x)] A_\mu(x)\, d^3x.$$

Using (1), (3), and (5), we transform $H(t)$ to the form

$$H(t) = -\sum \operatorname{Im}\{\varepsilon_{\alpha\beta} W_{mn}(t)\} M_{m\alpha, n\beta}, \tag{10}$$

where

$$W_{mn}(t) = \frac{e}{2} \int \bar{u}_m(x) \gamma_\mu u_n(x) A_\mu(x)\, d^3x,$$

and the matrix

$$\|\varepsilon_{\alpha\beta}\| = \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}.$$

It is clear from (10) that $H(t)$, as a linear combination of the operators (5), is an operator of an infinitesimal rotation of the space E_∞. Therefore the operator $S(t, t_0)$ will be a finite rotation of this space and, consequently, can be represented in the form (6).

The relations between the spinor and vector representations appear in field theory as relations between the interaction representation and the Heisenberg representation. This is seen from the well-known formula

$$\psi_H(x) = S^{-1}(t, t_0) \psi(x) S(t, t_0),$$

where $\psi_H(x)$ is the field operator in the Heisenberg representation. $\psi(x)$ is a linear combination of the quantities Γ_A, which, in view of (7), undergo a linear transformation V. Relation (8) shows how, by means of

solutions of the Heisenberg equations of motion for $\psi_H(x)$, one can construct the operator $S(t, t_0)$.

The established connection between the motion of an electron in an external field $A_\mu(x)$ and rotations of the space E_∞ makes it possible to outline a new approach to the solution of general problems of quantum field theory. Instead of determining the propagation function of an electron moving in an arbitrary external field $A_\mu(x)$, i.e. seeking the general solution of the Dirac equation with an arbitrary field, we can turn to the study of comparatively simple objects—orthogonal transformations of the space E_∞. It is true that along this path difficulties arise connected with the fact that not every rotation S can be obtained with the aid of relations (9) and (10). By varying the field $A_\mu(x)$ in these relations in all possible ways, we obtain only a certain class of such rotations. Thus there arises the problem of describing this class. One of the simplest conditions satisfied by the rotations S obtained in the indicated way is that they all commute with the charge operators Q. Of course, this is only a necessary, but by no means sufficient, condition.

In addition, we can pass from functional integration over the field $A_\mu(x)$ to integration over some subset of the elements of the group of rotations of the space E_∞.

In conclusion, we note that the connection considered here between the theory of Fermi fields and the theory of spinors of the space E_∞ rests only on the anticommutation relations (2), characteristic of Fermi statistics. Therefore the approach outlined here is easily carried over to any variant of the theory of Fermi fields, for example, to the theory of quasifields[5], and can also find application in problems of statistical physics.

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