

# ASYMPTOTIC FORMULAS FOR GENERALIZED LEGENDRE FUNCTIONS AND GEGENBAUER FUNCTIONS

1957

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**Abstract**

**Full Text**

**MATHEMATICAL PHYSICS**

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**ASYMPTOTIC FORMULAS FOR GENERALIZED LEGENDRE FUNCTIONS AND GEGENBAUER FUNCTIONS**

*(Presented by Academician N. N. Bogolyubov, 18 III 1957)*

Using the method set forth in the monograph <sup>(1)</sup>, one can find asymptotic expressions for the generalized Legendre functions  $P_l^m(\cos \varphi)$ ,  $Q_l^m(\cos \varphi)$ ,  $P_l^m(\operatorname{ch} \varphi)$ , and  $Q_l^m(\operatorname{ch} \varphi)$  for large values of  $l$  and fixed  $m$ . These expressions are valid over a wide range of variation of  $\varphi$ ; in particular, in contrast to the corresponding asymptotic expansions given in <sup>(2)</sup>, the greatest accuracy is obtained for values of  $\varphi$  close to zero.

The essence of the method is that the solution of the generalized Legendre differential equation is sought in the form of a product of two arbitrary functions. In the resulting equation we neglect a quantity that tends rapidly to zero as  $l$  increases. This makes it possible to determine the arbitrary functions exactly.

As is known, the functions  $P_l^m(\cos \varphi)$  and  $Q_l^m(\cos \varphi)$  satisfy the differential equation

$$u'' + \operatorname{ctg} \varphi u' + \left\{ l(l+1) - \frac{m^2}{\sin^2 \varphi} \right\} u = 0. \quad (1)$$

We shall seek the solution of this equation in the form <sup>(3)</sup>

$$u = f(\varphi)\psi(\varphi), \quad (2)$$

where  $f$  and  $\psi$  are unknown functions. Substituting this expression for  $u$  into the differential equation, we obtain

$$f'' + \frac{f'}{\varphi} + \left[ \left( l + \frac{1}{2} \right)^2 (1 + \varepsilon) - \frac{m^2}{\varphi^2} \right] f + \frac{f'}{\psi} \left[ -\frac{\psi}{\varphi} + \operatorname{ctg} \varphi \psi + 2\psi' \right] = 0, \quad (3)$$

where

$$\varepsilon = \frac{-\frac{1}{4} + \operatorname{ctg} \varphi \frac{\psi'}{\psi} + \frac{\psi''}{\psi} + \frac{m^2}{\varphi^2} - \frac{m^2}{\sin^2 \varphi}}{\left(l + \frac{1}{2}\right)^2}. \quad (4)$$

Equating to zero the bracket multiplying  $f'/\psi$ , we find

$$\psi(\varphi) = \left(\frac{\varphi}{\sin \varphi}\right)^{1/2}. \quad (5)$$

Substituting the found values of  $\psi$  and its derivatives into the expression for  $\varepsilon$ , we obtain

$$\varepsilon = \frac{(m^2 - \frac{1}{4}) \left(\frac{1}{\varphi^2} - \frac{1}{\sin^2 \varphi}\right)}{\left(l + \frac{1}{2}\right)^2}. \quad (6)$$

Whence it is seen that  $\varepsilon$  rapidly tends to zero for  $l \gg m$ , and as  $\varphi \rightarrow 0$

$$\varepsilon \rightarrow \frac{m^2 - 1/4}{3(l + 1/2)^2}.$$

Neglecting the quantity  $\varepsilon$  in equation (3), we obtain, for determining the function  $f$ , Bessel's equation

$$f'' + \frac{f'}{\varphi} + \left\{ (l + 1/2)^2 - \frac{m^2}{\varphi^2} \right\} f = 0,$$

whose solution is conveniently represented in the form

$$f(\varphi) = AJ_{-m}((l + 1/2)\varphi) + BN_{-m}((l + 1/2)\varphi), \quad (7)$$

where  $J_{-m}$  and  $N_{-m}$  are Bessel functions of the first and second kind;  $A$  and  $B$  are constants to be determined. Determining  $A$  and  $B$ , we find the desired asymptotic formulas:

$$P_l^m(\cos \varphi) = (l + 1/2)^m \left(\frac{\varphi}{\sin \varphi}\right)^{1/2} J_{-m}((l + 1/2)\varphi); \quad (8)$$

$$Q_l^m(\cos \varphi) = -\frac{\pi}{2} (l + 1/2)^m \left(\frac{\varphi}{\sin \varphi}\right)^{1/2} N_{-m}((l + 1/2)\varphi). \quad (9)$$

Analogously one obtains the formulas

$$P_l^m(\operatorname{ch} \varphi) = (l + 1/2)^m \left( \frac{\varphi}{\operatorname{sh} \varphi} \right)^{1/2} I_{-m}((l + 1/2)\varphi); \quad (10)$$

$$Q_l^m(\operatorname{ch} \varphi) = (-1)^m (l + 1/2)^m \left( \frac{\varphi}{\operatorname{sh} \varphi} \right)^{1/2} K_m((l + 1/2)\varphi), \quad (11)$$

where  $I_{-m}$  and  $K_m$  are Bessel functions of imaginary argument. In deriving formulas (10) and (11) in the exact equation we neglected the quantity

$$\varepsilon' = \frac{(m^2 - 1/4) \left( \frac{1}{\varphi^2} - \frac{1}{\operatorname{sh}^2 \varphi} \right)}{(l + 1/2)^2}.$$

For  $m = 0$ , formulas (8), (9), (10), and (11) become, respectively,

$$P_l(\cos \varphi) = \left( \frac{\varphi}{\sin \varphi} \right)^{1/2} J_0((l + 1/2)\varphi); \quad (12)$$

$$Q_l(\cos \varphi) = -\frac{\pi}{2} \left( \frac{\varphi}{\sin \varphi} \right)^{1/2} N_0((l + 1/2)\varphi); \quad (13)$$

$$P_l(\operatorname{ch} \varphi) = \left( \frac{\varphi}{\operatorname{sh} \varphi} \right)^{1/2} I_0((l + 1/2)\varphi); \quad (14)$$

$$Q_l(\operatorname{ch} \varphi) = \left( \frac{\varphi}{\operatorname{sh} \varphi} \right)^{1/2} K_0((l + 1/2)\varphi). \quad (15)$$

These formulas were obtained in the work (4).

From the expressions for  $\varepsilon$  and  $\varepsilon'$  it follows that for  $m = \pm 1/2$  the approximate equalities become exact. For example, from (8) one obtains (5)

$$P_l^{1/2}(\cos \varphi) = \left( \frac{2}{\pi \sin \varphi} \right)^{1/2} \cos(l + 1/2)\varphi; \quad (16)$$

$$P_l^{-1/2}(\cos \varphi) = \left( \frac{2}{\pi \sin \varphi} \right)^{1/2} \frac{\sin(l + 1/2)\varphi}{l + 1/2}. \quad (17)$$

Analogously, the remaining functions of order  $m = \pm 1/2$  are expressed through elementary functions.

In the derivation no assumption was made that  $l$  is a real number. Therefore the following asymptotic formulas are valid for the so-called conical functions, which often occur in mathematical physics:

$$P_{-1/2+i\lambda}(\cos \varphi) = \left( \frac{\varphi}{\sin \varphi} \right)^{1/2} I_0(\lambda\varphi); \quad (18)$$

$$Q_{-1/2+i\lambda}(\cos \varphi) = -\frac{\pi i}{2} \left( \frac{\varphi}{\sin \varphi} \right)^{1/2} I_0(\lambda\varphi) + \left( \frac{\varphi}{\sin \varphi} \right)^{1/2} K_0(\lambda\varphi); \quad (19)$$

$$P_{-1/2+i\lambda}(\operatorname{ch} \varphi) = \left( \frac{\varphi}{\operatorname{sh} \varphi} \right)^{1/2} J_0(\lambda\varphi); \quad (20)$$

$$Q_{-1/2+i\lambda}(\operatorname{ch} \varphi) = -\frac{\pi i}{2} \left( \frac{\varphi}{\operatorname{sh} \varphi} \right)^{1/2} H_0^{(2)}(\lambda\varphi). \quad (21)$$

A function closely related to the generalized Legendre function  $P_l^m(\cos \varphi)$  is the Gegenbauer function  $C_l^\nu(\cos \varphi)$ , which, for integral  $\nu$ , is defined as the coefficient of  $h^\nu$  in the expansion of  $(1 - 2h \cos \varphi + h^2)^{-\nu}$  in increasing powers of  $h$ . As is known, the Gegenbauer function is connected with the generalized Legendre function by the relation

$$C_l^\nu(\cos \varphi) = \frac{\Gamma(2\nu + l)\Gamma(\nu + 1/2)}{\Gamma(2\nu)\Gamma(l + 1)} \left\{ \frac{\sin \varphi}{2} \right\}^{1/2-\nu} P_{l+\nu-1/2}^{1/2-\nu}(\cos \varphi). \quad (22)$$

Hence one obtains the asymptotic equality

$$C_l^\nu(\cos \varphi) = \frac{\Gamma(2\nu + l)\Gamma(\nu + 1/2)}{\Gamma(2\nu)\Gamma(l + 1)} \left( \frac{l + \nu}{2} \right)^{1/2-\nu} \frac{\sqrt{\varphi}}{\sin^\nu \varphi} J_{\nu-1/2}((l + \nu)\varphi), \quad (23)$$

which, for  $\nu = 1$ , passes, as was to be expected, into the exact formula

$$C_l^1(\cos \varphi) = \frac{\sin(l + 1)\varphi}{\sin \varphi}. \quad (24)$$

Formula (23) can be simplified if one again uses the condition  $l \gg \nu$ :

$$C_l^\nu(\cos \varphi) = \left( \frac{l}{2} \right)^{\nu-1/2} \frac{\sqrt{\pi}}{\Gamma(\nu)} \frac{\sqrt{\varphi}}{\sin^\nu \varphi} J_{\nu-1/2}((l + \nu)\varphi). \quad (25)$$

It is easy to verify that for  $\nu = 1/2$  formula (25) passes into (12). Calculations carried out with the aid of tables <sup>(6)</sup> showed good accuracy already for  $l = 10$ .

In practical computations it must be borne in mind that here the definitions of the generalized Legendre functions adopted are those given by Hobson (2).

Formulas (8), (12), (18), and (25) represent a special case of the asymptotic formula for the Gauss hypergeometric function for large values of the parameter  $\lambda$ :

$$\begin{aligned}
 {}_2F_1\left(\alpha - \lambda, \beta + \lambda, \frac{\alpha + \beta + 1}{2}, \sin^2 \frac{\varphi}{2}\right) &= \\
 &= \frac{2^{\frac{\alpha + \beta - 1}{2}} \Gamma\left(\frac{\alpha + \beta + 1}{2}\right)}{\left(\lambda + \frac{\beta - \alpha}{2}\right)^{\frac{\alpha + \beta - 1}{2}} \sin^{\frac{\alpha + \beta}{2}} \varphi} \frac{\sqrt{\varphi}}{\varphi} J_{\frac{\alpha + \beta - 1}{2}}\left(\left(\lambda + \frac{\beta - \alpha}{2}\right) \varphi\right).
 \end{aligned}$$

For Whittaker's degenerate hypergeometric function  $M_{\lambda, \mu}(z)$ , for large values of  $\lambda$ , the following asymptotic formula is valid:

$$M_{\lambda, \mu} = \Gamma(2\mu + 1) \lambda^{-\mu} \sqrt{z} J_{2\mu}(2\sqrt{\lambda z}).$$

An estimate of the remainder term in all the formulas can be obtained by means of the Liouville-Steklov method.

I consider it my duty to express my gratitude to Prof. A. A. Sokolov for supervising this work.

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Received  
12 III 1957

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*Note: Figure translations are in progress. See original paper for figures.*

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