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Abstract

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MATHEMATICS

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INVESTIGATION OF A NONLINEAR SYSTEM OF THREE DIFFERENTIAL EQUATIONS

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In the present paper a special case of a system of three equations of Aizerman type (1) is studied. We consider the system of equations

$$\frac{dx}{dt} = f_1(x) + a_{12}y + a_{13}z,$$

$$\frac{dy}{dt} = a_{21}x + a_{22}y + a_{23}z, \tag{1}$$

$$\frac{dz}{dt} = a_{31}x + a_{32}y + a_{33}z$$

under the condition $a_{22} + a_{33} = 0$. By a change of variables this system can be brought to the form

$$\frac{dx}{dt} = y - f(x), \qquad \frac{dy}{dt} = z - x, \qquad \frac{dz}{dt} = -ax - bf(x). \tag{2}$$

Concerning the function $f(x)$, we shall assume that it satisfies the Lipschitz condition and the generalized Hurwitz conditions:

$$\frac{f(x)}{x} > a + b\frac{f(x)}{x} > 0 \quad \text{for } x \neq 0; \qquad f(0) = 0. \tag{3}$$

A special case of system (2) under conditions (3) was considered by us in the note (2).

Let $\varphi(p, t)$ denote the trajectory of system (2) which at $t = 0$ passes through the point p of phase space. Using qualitative methods, one can prove the following theorems.

Theorem 1. Any positive semitrajectory of system (2) lying entirely in one of the half-spaces $x \geq 0$ or $x \leq 0$ tends to the origin.

Theorem 2. Suppose the inequalities $a > 0,\ 0 \leq b < 1,\ a + b \geq 1$ are satisfied, and suppose the point p lies in the plane $x = 0$; then the trajectory $\varphi(p, t)$ of system (2) for $t > 0$ intersects the plane $x = 0$.

Theorem 3. Suppose that the inequalities $a > 0,\ 0 \leq b < 1,\ a + b \geq 1$ are satisfied. Let the point p, distinct from $x = y = z = 0$, lie in the plane $x = 0$, and let $t_1 > 0$ be the first moment after $t = 0$ at which the trajectory $\varphi(p, t)$ intersects the plane $x = 0$; let t_2 $(t_2 > t_1)$ be the first moment after t_1 at which $\varphi(p, t)$ intersects the plane $x = 0$.

Then one of two possibilities holds:

I. Either $y(\varphi(p, t_1)) > 0,\ z(\varphi(p, t_1)) > 0$, and then $y(\varphi(p, t_2)) < 0,\ z(\varphi(p, t_2)) < 0$.

II. Or $y(\varphi(p, t_1)) < 0,\ z(\varphi(p, t_1)) < 0$, and then $y(\varphi(p, t_2)) > 0,\ z(\varphi(p, t_2)) > 0$.

For systems of the Aizerman type it often proves possible to construct a Lyapunov function of the form "an integral of the nonlinearity plus a quadratic form in the coordinates of phase space" (see, for example, (3)). To construct functions of this kind it is convenient to use the following device.

Consider a nonlinear system of differential equations of the form

$$\frac{dx_1}{dt} = \sum_{j=1}^{n} a_{1j} x_j + f(x_k), \qquad \frac{dx_i}{dt} = \sum_{j=1}^{n} a_{ij} x_j \quad (i = 2, 3, \dots, n) \tag{4}$$

and, along with it, the system of linear equations with constant coefficients

$$\frac{dx_1}{dt} = \sum_{j=1}^{n} a_{1j} x_j + F x_k, \qquad \frac{dx_i}{dt} = \sum_{j=1}^{n} a_{ij} x_j \quad (i = 2, 3, \dots, n). \tag{5}$$

It is easy to prove the following lemmas.

Lemma 1. If the quadratic form

$$v = W(x_1, x_2, \dots, x_n) + \frac{\mu}{2} F x_k^2, \tag{6}$$

where F is a constant number and W is a quadratic form in the variables $x_1, x_2, \dots, x_n$ with coefficients independent of F, is positive definite for all $F \in (\gamma\delta)$, then the function

$$v_1 = W(x_1, x_2, \dots, x_n) + \mu \int_0^{x_k} f(x)\, dx \tag{7}$$

is also positive definite for any continuous $f(x)$ satisfying the conditions

$$f(0) = 0, \qquad \gamma < \frac{f(x)}{x} < \delta \quad \text{for } x \neq 0. \tag{8}$$

Suppose that the quadratic form W has coefficients independent of F. Denote by $\dot{v}$ and $\dot{v}_1$ the time derivatives of the functions v and v_1, taken along the systems (5) and (4), respectively.

Lemma 2. If for all $F \in (\gamma\delta)$ $\dot{v} \leq 0$, then for any continuous $f(x)$ satisfying the conditions (8), $\dot{v}_1 \leq 0$.

It follows from these lemmas that if for the system (5) there exists a Lyapunov function of the form (6) for all $F \in (\gamma\delta)$, then for the system (4) there also exists a Lyapunov function for any continuous $f(x)$ satisfying the conditions (8), and this function has the form (7). It is further clear that if for the system (5) one cannot specify a function v of the form (6) having a negative time derivative $\dot{v}$ for all $F \in (\gamma\delta)$, then for the system (4) there does not exist a Lyapunov function of the form (7). Thus the question of the existence and construction, for the system (4), of a Lyapunov function of the special form indicated above reduces to the question of the existence and construction, for the system (5), of a Lyapunov function of the form (6).

Using this method for constructing a Lyapunov function, and also using the results of papers (^{5,6}), one can prove the following theorem.

Theorem 4. *If one of the following three conditions is satisfied:*

$$a < 0,\ b > 0; \quad 2)\ a = 0,\ 0 < b < 1; \quad 3)\ a > 0,\ b < 0,\ a^2 + b(1-b)^2 \leq 0,$$

then the zero solution of system (2) is asymptotically stable in the large for any function $f(x)$ satisfying the generalized Hurwitz conditions (3).

In the case when $a > 0,\ a^2 + b(1-b)^2 > 0$, such an assertion cannot be made; however, one can indicate conditions sufficient for asymptotic stability in the large of the zero solution of system (2). Namely, the following theorems hold.

Theorem 5. *Suppose that the inequalities $a > 0$, $a^2 + b(1-b)^2 > 0$, and the generalized Hurwitz conditions (3) are satisfied. Suppose, moreover, that for all real $x \neq 0$ the inequalities*

$$0 < xf(x) - \frac{a}{1-b}x^2 \leq \frac{1-b}{a}x^2; \tag{9}$$

hold; then the zero solution of system (2) is asymptotically stable in the large.

The condition of Theorem 5 narrows the sector of variation of the function $f(x)$ given by the generalized Hurwitz conditions (3). The following sufficient condition for stability in the large is free of this drawback, but instead it restricts the limits of variation of the derivative of $f(x)$.

Theorem 6. *Suppose that the inequalities* $a > 0$, $a^2 + b(1-b)^2 > 0$, *and the generalized Hurwitz conditions (3) are satisfied. Suppose, moreover, that for all real* x *the function* $f(x)$ *is differentiable and satisfies the inequality*

$$\frac{df}{dx} > \frac{a}{1-b}. \tag{10}$$

Then the zero solution of system (2) is asymptotically stable in the large.

In the cases when $a > 0$, $a^2 + b(1-b)^2 > 0$, i.e., in those cases when the zero solution of system (2) proves to be asymptotically stable in the large not for all $f(x)$, the qualitative picture of the behavior of the solution is clarified in greater detail.

Theorem 7. *Let the inequalities* $a > 0$, $a^2+b(1-b)^2 > 0$ *be satisfied. Suppose, further, that there exist numbers* $\varepsilon > 0$ *and* $x_0 > 0$ *such that*

$$\frac{df(x)}{dx} - \frac{a}{1-b} > \varepsilon, \quad \text{for } |x| \geq x_0. \tag{11}$$

Then there exists an $M > 0$ *such that, for any point* p *of the phase space, one can indicate a* T_p *such that, for* $t \geq T_p$, *along the trajectory* $\varphi(p,t)$ *of system (2), the inequalities*

$$|x| < M, \quad |y| < M, \quad |z| < M. \tag{12}$$

hold.

Theorem 8. *Let the conditions* $a > 0$, $0 \leq b < 1$, $a+b \geq 1$ *be satisfied. Suppose there exist positive numbers* x_0 *and* ε *such that condition (11) is satisfied. Then, in order that the zero solution of system (2) be asymptotically stable in the large, it is necessary and sufficient that system (2) have no periodic motions.*

The following theorem clarifies the question of the structure of the limit sets of those trajectories of system (2) which do not tend to the origin as $t \to \infty$.

Theorem 9. Suppose the inequalities $a > 0$, $a + b \geqslant 1$, $0 \leqslant b < 1$ are satisfied; suppose there exist numbers $x_0 > 0$, $\varepsilon > 0$ such that condition (11) is satisfied. Then any trajectory of system (2) which does not tend to the origin as $t \to +\infty$ has in its ω-limit set the oscillatory regime [6].

In conclusion we indicate two conditions sufficient for the existence in system (2) of periodic solutions distinct from the zero solution.

Theorem 10. Suppose the inequalities $a > 0$, $0 \leqslant b < 1$, $a+b \geqslant 1$ are satisfied. Suppose the conditions

$$hx \leqslant f(x) - \frac{a}{1-b}x \quad \text{for } 0 \leqslant x \leqslant x_1; \tag{13}$$

$$0 < f(x) - \frac{a}{1-b}x < \lambda \quad \text{for } x \geqslant x_2, \tag{14}$$

are satisfied, where the number $h > \frac{1-b}{a}$, and the numbers $\lambda,\ x_1,\ x_2 - x_1$ are positive and sufficiently small. Suppose, moreover,

$$f(x) = -f(-x). \tag{15}$$

Then system (2) has a periodic solution.

Theorem 11. Suppose the inequalities $a > 0,\ a^2 + b(1-b)^2 > 0$ are satisfied. Suppose, moreover, that conditions (13), (14), (15) and

$$\frac{df}{dx} > \frac{a}{1-b} \quad \text{for } 0 \leqslant x < x_1; \tag{16}$$

$$f(x) - \frac{a}{1-b}x < f(x_1) - \frac{a}{1-b}x_1 \quad \text{for } x > x_1. \tag{17}$$

are satisfied. Under these conditions, as above, $h > \frac{1-b}{a}$, and the numbers $\lambda,\ x_1,\ x_2 - x_1$ are sufficiently small. Then system (2) has a periodic motion.

It follows from Theorem 11 that, for $a > 0,\ a^2 + b(1-b)^2 > 0$, the zero solution of system (2) may fail to be stable in the large, despite the fulfillment of the generalized Hurwitz conditions. Hence, and from Theorem 4, the following theorem follows.

Theorem 12. In order that the zero solution of system (2) be asymptotically stable in the large for every function $f(x)$ satisfying the generalized Hurwitz conditions, it is necessary and sufficient that one of the following three conditions be fulfilled:

1) $a < 0,\ b > 0$;
2) $a = 0,\ 0 < b < 1$;
3) $a > 0,\ b < 0,\ a^2 + b(1-b)^2 \leqslant 0$.

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Note: Figure translations are in progress. See original paper for figures.

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