Soviet-era science, translated into English

ON THE CONVERGENCE OF THE SOLUTION OF A DIFFERENCE EQUATION TO THE SOLUTION OF A DIFFERENTIAL EQUATION

1957

SovietRxiv

View the original and related papers at https://sovietrxiv.org/items/ru-195701.69638

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

SovietRxiv

Abstract

Full Text

MATHEMATICS

A. I. ZHUKOV

ON THE CONVERGENCE OF THE SOLUTION OF A DIFFERENCE EQUATION TO THE SOLUTION OF A DIFFERENTIAL EQUATION

(Presented by Academician M. V. Keldysh on 30 V 1957)

The main result of the present note reduces to the existence of functional spaces in which the conditions for convergence of the solution of a difference equation to the solution of the corresponding differential equation are considerably broader than the commonly known stability conditions (for example, the Courant condition). Such spaces are, in particular, the spaces of generalized functions over the fundamental spaces, considered by I. M. Gel' fand and G. E. Shilov ([1,2]), consisting of entire functions.

1. We shall denote by Z the space consisting of all entire analytic functions $\varphi(z)$ of the complex variable z with convergence uniform in every bounded domain. For all operators considered below in Z we shall assume that their domains of definition satisfy the following conditions:

a) they are linear manifolds;

b) they contain all polynomials and all exponentials e^{az} (a is any complex number);

c) together with each function $\varphi(z)$ they contain its derivative $\varphi'(z)$ and all its shifts $\varphi(z+h)$ (h is any complex number).

The **shift operator** E^h is the operator taking the function $\varphi(z)$ into $\varphi(z+h)$.

2. A linear continuous operator A is called **constant** if it commutes with any shift operator, i.e. if for any complex h $E^hA = AE^h$. Every constant operator A can be expanded in a series in powers of the differentiation operator D:

$$A = \sum_{k=0}^{\infty} \frac{a_k}{k!} D^k.$$

The complex coefficients a_k are the **moments** of the operator A; they may be defined as follows: $a_k = Az^k$ at $z = 0$. The function

SovietRxiv

$$u(z) = \sum_{k=0}^{\infty} \frac{a_k}{k!} z^k$$

is an entire function; we call it the **characteristic function** of the operator A and write $A = u(D)$.

All constant operators commute with one another; the characteristic function of the product of two operators is equal to the product–

to the introduction of their characteristic functions. The operator A is invertible if and only if there exists an operator B such that

$$A = e^B = \sum_{k=0}^{\infty} \frac{1}{k!} B^k.$$

In order that a sequence of constant operators converge to some constant operator, it is necessary and sufficient that the sequence of their characteristic functions converge uniformly in every bounded domain.

If the zeroth moment a_0 of the operator A is different from zero, the logarithm of its characteristic function $u(z)$, in a neighborhood of the point $z = 0$, can be expanded in a power series

$$\ln u(z) = \sum_{k=0}^{\infty} \frac{\chi_k}{k!} z^k.$$

The coefficients χ_k are the **semi-invariants** of the operator A. (For the relations between moments and semi-invariants, see, for example, ([3]), p. 210.)

3. Let A be a constant operator, and t a real parameter. The equation

$$\frac{\partial \varphi}{\partial t} = A\varphi \tag{1}$$

with the initial condition $\varphi(t_0, z) = \varphi_0(z)$ has the solution

$$\varphi = e^{(t-t_0)A} \varphi_0$$

in the domain where the corresponding operators are defined. If A is a polynomial in the differentiation operator,

$$A = c_0 + \frac{c_1}{1!} D + \dots + \frac{c_p}{p!} D^p,$$

SovietRxiv

then the operator $e^{(t-t_0)A}$ has semi-invariants $\chi_k = (t-t_0)c_k,\ k = 0, 1, \dots, p$; for $k > p$, $\chi_k = 0$. Questions of existence and uniqueness of the solution of equation (1) for this case were investigated by I. M. Gelfand and G. E. Shilov ([1,2]).

4. Let the constant operator F_τ, depending on the real parameter τ, satisfy the following requirements:

a) its zeroth moment $a_0(\tau)$ does not vanish;

b) as $\tau \to 0$, the ratios χ_k/τ tend to finite limits c_k;

c) for $k > p \geq 0$, $c_k = 0$.

Then, as $n \to \infty$ and $\tau = \dfrac{t-t_0}{n}$, the sequence of operators F_τ^n converges to the operator

$$\exp\left[(t-t_0)\left(c_0 + \frac{c_1}{1!}D + \dots + \frac{c_p}{p!}D^p\right)\right],$$

which takes the function $\varphi_0(z)$ into the function $\varphi(t,z)$ satisfying the equation

$$\frac{\partial\varphi}{\partial t} = c_0\varphi + \frac{c_1}{1!}\frac{\partial\varphi}{\partial z} + \dots + \frac{c_p}{p!}\frac{\partial^p\varphi}{\partial z^p} \tag{2}$$

and the initial condition $\varphi(t_0, z) = \varphi_0(z)$.

5. All the definitions and theorems given above carry over directly to spaces of linear continuous functionals over linear manifolds in Z. This, in particular, applies to the sim-

the spaces of generalized functions over the basic spaces Z_a^q mentioned above, consisting of entire functions satisfying inequalities of the form

$$|\varphi(x+iy)| < ae^{b|y|^q - c|x|^q}$$

$(q > 1;\ a, b, c > 0)$.

6. The conditions of item 4 are satisfied, in particular, by many difference operators used for the numerical integration of equations of the form (2). Suppose, for example, that a numerical integration formula has the form

$$\varphi(t+\tau, z) = \sum_m b_m \varphi(t, z+mh), \tag{3}$$

where m runs through some finite set of indices, and the coefficients b_m depend only on the real parameters τ and h (the mesh steps). Formula (3) corresponds to the difference operator

SovietRxiv

$$G_{\tau,h} = \sum_m b_m E^{mh},$$

whose moments are

$$\alpha_k = h^k \sum_m m^k b_m.$$

Let $h = h(\tau)$, with $h \to 0$ as $\tau \to 0$; then the operator G will depend only on τ. If, under this condition, it satisfies the conditions of item 4, we arrive at a differential equation whose solution, in the limit as $\tau \to 0$, is given by formula (3).

This equation depends, generally speaking, on the choice of the function $h(\tau)$. There exist difference equations for which the limits of the ratios χ/τ do not depend on the form of the function $h(\tau)$ (in the terminology of I. M. Gel' fand, they are called flexible); for them the limiting theorem of item 4 is always valid.

In any case, the conditions of item 4 are considerably weaker than the usual stability conditions, which are essentially connected with the signs of the even semi-invariants. In particular, unstable numerical integration formulas may also give the correct solution (in the limit). The reason for this lies, of course, in the nature of the topology of the spaces we have chosen.

The practical use of unstable formulas is limited by the circumstance that functions arising in the process of integration may grow without bound. However, in the space of generalized functions one can introduce a process of "smoothing," by means of which any sequence converging to a functional of function type can be transformed into a bounded sequence of continuous functions converging to the same limit. Combining integration by a formula of type (3) with such "smoothing" (not necessarily at every step), one can obtain a process quite acceptable from the practical point of view. Since the basic formula need not be stable, conditions on the steps (of the Courant-condition type) become immaterial.

Department of Applied Mathematics
V. A. Steklov Mathematical Institute
Academy of Sciences of the USSR

Received
28 V 1957

CITED LITERATURE

1. I. M. Gel' fand, G. E. Shilov, *Uspekhi Mat. Nauk*, **8**, no. 6 (58), 3 (1953).

2. I. M. Gel' fand, G. E. Shilov, *Dokl. Akad. Nauk SSSR*, **102**, No. 6 (1955).

3. H. Cramér, *Mathematical Methods of Statistics*, Moscow, 1948.

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.