

ON THE GROWTH OF THE NUMBER OF COMPLETELY CRITICAL CYCLIC FIELDS OF DEGREE $\backslash(I^h\backslash)$

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Abstract

Full Text

MATHEMATICS

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ON THE GROWTH OF THE NUMBER OF COMPLETELY CRITICAL CYCLIC FIELDS OF DEGREE l^h

(Presented by Academician I. M. Vinogradov on 19 XI 1956)

This note considers the question of the distribution of completely critical cyclic fields of degree l^h , whose discriminants do not exceed a given bound, where l is a prime*; h is any positive integer. A field K will be called **completely critical** if all prime divisors of the discriminant of K are completely critical prime numbers of the field K .

There exists, obviously, an infinite set of completely critical fields. For example, even the number of completely critical cyclic fields of degree l^h whose discriminants consist of only one prime divisor is infinite, since there are infinitely many prime numbers of the form $p \equiv 1 \pmod{l^h}$. In particular, the completely critical fields include cyclic fields having degree l .

Relying on group-theoretic considerations and on the Kronecker-Weber theorem on abelian fields over the field of rational numbers, one can derive a general expression for the discriminant of abelian fields of degree l^α , α an integer.

Let K_1 be an abelian field of degree l^α over the field of rational numbers; G the Galois group of the field K_1 . Let G be a group of type

$$(l^h, \dots, l^h; l^{h_1}, \dots, l^{h_1}, \dots, l^{h_\nu}, \dots, l^{h_\nu}), \quad (1)$$

where the component l^h occurs k times, $l^{h_1-k_1}$ times, and so on, $l^{h_\nu-k_\nu}$ times,

$$kh + k_1h_1 + \dots + k_\nu h_\nu = \alpha; \quad h > h_1 > \dots > h_\nu \geq 1.$$

Theorem 1. *The discriminant D of an abelian field K_1 of type (1) is represented in the form*

$$D = \left(\prod_{i_1} p_{i_1}^{(h)} \right)^{H(1-\frac{1}{l^h})} \left(\prod_{i_2} p_{i_2}^{(h-1)} \right)^{H(1-\frac{1}{l^{h-1}})} \dots \left(\prod_{i_h} p_{i_h}^{(1)} \right)^{H(1-\frac{1}{l})},$$

where $H = l^\alpha$; $p_{i_\mu}^{(h+1-\mu)}$ are distinct prime numbers of the form $1 + tl^{h+1-\mu}$ ($\mu = 1, 2, \dots, h$); $i_\mu \geq 0$, but $i_1 \geq 1$.

As a consequence of this theorem, one obtains the discriminants of the fields that were considered in the work (2).

Theorem 2. *The discriminant of a completely critical abelian field of degree l^α has the form*

$$(p_1 p_2 \cdots p_m)^{H(1-\frac{1}{l^h})},$$

where $H = l^\alpha$; $p_i \equiv 1 \pmod{l^h}$ ($i = 1, 2, \dots, m$) are distinct prime numbers, $m \geq 1$.

* It is assumed that l is a noncritical prime number.

Theorem 3. The discriminant of a completely critical cyclic field of degree l^h is

$$(p_1 p_2 \cdots p_m)^{l^h - 1}, \quad (2)$$

where $p_i \equiv 1 \pmod{l^h}$ ($i = 1, 2, \dots, m$) are distinct prime numbers, $m \geq 1$.

Theorem 4. The number of completely critical cyclic fields of degree l^h having discriminant (2) is equal to $\varphi(l^h)^{m-1}$.

For the proof, see paper (3).

Theorem 5. The number N of completely critical cyclic fields of degree l^h whose discriminants do not exceed x^{l^h-1} is given by the asymptotic formula

$$N = \lambda x + O\left(x^{1-\frac{1}{\varphi(l^h)}+\varepsilon}\right), \quad (3)$$

where $\lambda > 0$ is a constant depending only on the structural properties of the fields under consideration; $\varepsilon > 0$ is an arbitrary number.

The proof, as in paper (2), is carried out by the classical method of the distribution of prime numbers, using results from the theory of Dirichlet L -functions (1). The function considered for this purpose,

$$f(s) = \prod_{p \equiv 1 \pmod{l^h}} \left(1 + \frac{\varphi(l^h)}{p^s}\right), \quad s = \sigma + it,$$

is regular in the half-plane $\sigma > 1/2$ and has there a single pole of first order at $s = 1$, since in the representation

$$f(s) = \prod_{\chi} L(s; \chi) \cdot \prod_{p \equiv 1(l^h)} \left(1 - \frac{1}{p^s}\right)^{\varphi(l^h)} \left(1 + \frac{\varphi(l^h)}{p^s}\right) \cdot \prod_{p \not\equiv 1(l^h)} \left(1 - \frac{1}{p^{fs}}\right)^g$$

the function defined by the first product on the right is regular in the whole plane of complex numbers, except for a simple pole at $s = 1$, while the last two products converge absolutely and uniformly for $\sigma > 1/2$. Here χ runs through all Dirichlet characters (mod l^h); $L(s; \chi)$ is the Dirichlet function of the character χ ; $fg = \varphi(l^h)$; f is the exponent to which p belongs (mod l^h).

The constant λ in formula (3) is represented by the expression

$$\lambda = \frac{1}{2} \left(1 - \frac{1}{l}\right) \prod_{p \equiv 1(l^h)} \left(1 - \frac{1}{p}\right)^{\varphi(l^h)} \left(1 + \frac{\varphi(l^h)}{p}\right) \cdot \prod_{p \not\equiv 1(l^h)} \left(1 - \frac{1}{p^f}\right)^g \lim_{s \rightarrow 1} \prod_{\chi \neq \chi_0} L(s; \chi),$$

which, by Dirichlet's theorem on the distribution of prime numbers in progressions, is nonzero; moreover, λx is the residue of the function

$$\frac{f(s)}{s(s+1)} x^s$$

at the point $s = 1$, integrated according to the Riemann-Hadamard construction.

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Note: Figure translations are in progress. See original paper for figures.

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