A GENERALIZATION OF THE PROBLEM OF THE EXTREME POINT OF INTERSECTION OF AN AXIS WITH A CONVEX POLYHEDRON

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Abstract

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A GENERALIZATION OF THE PROBLEM OF THE EXTREME POINT OF INTERSECTION OF AN AXIS WITH A CONVEX POLYHEDRON

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L. V. Kantorovich ([1]) considered several extremal problems of a production-economic nature, for the solution of which he developed the method of *resolving multipliers*. Subsequently the method was applied in works ([2]-[6], [14]), devoted to the detailed study of some of the problems considered earlier, as well as to the investigation of new questions. In works ([7]-[9]), by the method of resolving multipliers, a geometric problem was solved to which all the preceding and some other problems reduce. However, in solving a number of concrete questions (in particular, some problems from ([1], [3])) by reduction to this geometric problem, the specificity of the questions is not fully used, and the resulting solutions are more laborious than under a direct application of the method of resolving multipliers.

In the present paper a more general geometric problem is considered, solved by the method of resolving multipliers; in reduction to this problem the specificity of concrete questions is reflected more fully, as a result of which the solutions obtained (in any case for the problems known to us) differ little from those obtained by a direct application of the method of resolving multipliers. The problem is illustrated by two examples.

Basic problem. In the real n-dimensional space R_n there are given finite sets of points

$$A_i = \{a_{ik}\}_{k=1,2,\dots,l_i} \quad (i = 1, 2, \dots, m)$$

and points c and d. Consider the polyhedron

$$M = \sum_i M_i{}^*,$$

where M_i is the convex hull of the set A_i, and the axis

$$B = \{b = c + \lambda d\}_{\lambda \in (-\infty, +\infty)}.$$

It is required:

1. To establish whether B and M have common points.
2. If $B \cap M \neq \Lambda$, then: a) determine λ_0, equal to the maximum of those λ for which $b = c + \lambda d \in M$; b) in the sets A_i, indicate the vertices of those simplexes inside which lie (respectively) points a_{i0} such that the extreme point of intersection of the axis B with the polyhedron M—the point

$$b_0 = c + \lambda_0 d = \sum_i a_{i0};$$

c) find the hyperplane H_0 strictly separating the ray

$$L(b_0, b_0 + d) = \{b = b_0 + \lambda d\}_{\lambda+0}$$

from the polyhedron M (the polyhedron M is situated on one side of H_0, $L(b_0, b_0 + d)$ on the other, and

$$L(b_0, b_0 + d) \cap H_0 = \Lambda).$$

3. If $B \cap M = \Lambda$, then find the hyperplane H_0 strictly separating the axis B from the polyhedron M.

The problem formulated will hereafter be called the problem of the extreme point of intersection of the axis B with the polyhedron M, or simply the problem of

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$$^* \sum_i M_i = \left\{ a = \sum_i a_i;\ a_i \in M_i \right\},$$

i.e. this is the Minkowski sum of sets.

extreme point. In ([7]-[9]) a special case of this problem was considered, corresponding to the value $m = 1$. The general problem can be reduced to this case, since the polyhedron M is the convex hull of the set

$$A = \sum_i A_i.$$

However, if in the original problem there are

$$N = \sum_i l_i$$

points, then in the new one their number is

$$N' = \prod_i l_i,$$

which considerably increases the laboriousness of the computations.

In the problem of the extreme point two dual problems are combined: one is connected with determining the extreme point b_0, the other with finding the separating hyperplane H_0. The method of resolving multipliers consists in reducing the first problem to the second. Therefore, in our case, for a complete solution of the problem it is sufficient to find the separating hyperplane H_0, which we call resolving. To compute the equation of the resolving hyperplane, the algorithm described in ([7],[9]) may be used with some modifications.

We give examples of reducing questions to the problem of the extreme point.

The problem of loading equipment*. There are m different machines, each of which can perform n kinds of work. It is known that the productivity (the amount of work performed per unit time) of the i-th ($i = 1, 2, \dots, m$) machine in performing work of the j-th ($j = 1, 2, \dots, n$) kind is α_{ij} units**. It is required to draw up a plan for distributing the work, i.e., to find numbers h_{ij} $\left(h_{ij} \geqslant 0,\ \sum_j h_{ij} \leqslant 1\right)$, indicating what part of the working time the i-th machine must perform work of the j-th kind, so that:

1) the necessary assortment is maintained, i.e. the numbers $z_j = \sum_i \alpha_{ij} h_{ij}$, expressing the quantities of work of different kinds performed by the entire aggregate of machines per unit time, are in the given ratio

$$\frac{z_1}{k_1} = \frac{z_2}{k_2} = \dots = \frac{z_n}{k_n};$$

2) the common value z of these ratios, expressing the aggregate productivity under the given plan, has the greatest possible value.

A work plan satisfying conditions 1) and 2) is called **optimal**.

The method of resolving multipliers reduces the stated problem, in which it is required to determine mn unknowns, to the problem of finding n numbers (resolving multipliers) $t_1, t_2, \dots, t_n$ ($t_j \geqslant 0$), for which the quantity

$$\sum_i p_i \Big/ \sum_j t_j k_j, \quad \text{where } p_i = \max_j t_j \alpha_{ij},$$

has the least possible value. This last value coincides with the maximal aggregate productivity. The numbers t_j may be interpreted as conditional labor requirements of the different kinds of work, and the numbers p_i as conditional productivities of the machines under the conditions of the given problem.

In order to translate the problem under consideration into geometric language, we shall characterize the productivity of the i-th machine under each plan $\|h_{ij}\|$ by a point $a_i \in R_n$, whose j-th coordinate is equal to the productivity under this plan of the i-th machine in the j-th kind of work. Then the total productivity is characterized by the point

$$a = \sum_i a_i.$$

The points a_i corresponding to all possible work plans of the i-th machine evidently fill

* This is one of the problems considered in ([1]).

** Each kind of work may be measured in its own unit.

simplex M_i with vertices corresponding to cases where the i-th machine tool all the time performs work of one kind or does not work at all. The points a corresponding to all possible aggregate plans fill the polyhedron

$$M = \sum_i M_i.$$

Plans under which the required assortment is fulfilled correspond to the points of the axis $B = \{b = \lambda d\}$, where $d = (k_1, k_2, \dots, k_n)$. In this case the aggregate productivity for each such plan is equal to λ. Hence it is clear that the optimal plans will be precisely those plans which correspond to the extreme point of intersection of the axis B with the polyhedron M.

Thus the question is reduced to the problem of an extreme point. In the present case one always has $B \cap M \neq \Lambda$, since B and M contain the origin. The direction coefficients of the separating hyperplane H_0 are the resolving multipliers, i.e. they express the conditional labor requirements of the various kinds of work, while the constant terms in the equations of hyperplanes parallel to H_0 and supporting the simplexes M_i express the conditional productivities of the corresponding machine tools.

The transportation problem*. There are m production points (p. pr.) of a homogeneous product and n consumption points (p. potr.) for this product. Given are the productivities p_i $(i = 1, 2, \dots, m)$ at each p. pr., the demands r_j $(j = 1, 2, \dots, n)$ at each p. potr. $\left(\sum_i p_i = \sum_j r_j\right)$, and the costs c_{ij} of transporting one unit of the product from the i-th p. pr. to the j-th p. potr. It is required to draw up a transportation plan, i.e. to find numbers $h_{ij} \geqslant 0$ expressing the quantity of product transported from the i-th p. pr. to the j-th p. potr., such that, in doing so:

$$1)\ \sum_j h_{ij} = p_i, \quad 2)\ \sum_i h_{ij} = r_j; \quad 3)\ \text{the total transportation costs}$$

$$z = \sum_{i,j} c_{ij} h_{ij}$$

are minimal.

A transportation plan satisfying conditions 1), 2), and 3) is called **optimal**.

The method of resolving multipliers reduces the posed problem to finding n numbers $u_1, u_2, \dots, u_n$ such that the quantity

$$\sum_i v_i p_i - \sum_j u_j r_j,$$

where

$$v_i = \min_j (u_j + c_{ij}),$$

has the greatest possible value. This value coincides with the minimal transportation costs. The numbers u_j, v_i may be interpreted as the potentials of the corresponding points (3). Goods should be sent only along routes for which the tension is

$$v_i - u_j = c_{ij}.$$

To translate the posed problem into geometric language, the shipments from the i-th p. pr. under each plan $\|h_{ij}\|$ will be characterized by a point $a_i \in R_{n+1}$, whose j-th ($j = 1, 2, \dots, n$) coordinate is equal to the quantity of product transported under this plan from the i-th p. pr. to the j-th p. potr., while the $(n+1)$-st coordinate is equal to the costs required for this. Then the aggregate transportation plan is characterized by the point

$$a = \sum_i a_i.$$

The points a_i,

* For the first time, apparently, this problem is singled out from a broad range of questions of the organization of transportation in the mathematical work (10), but it does not find a complete solution there. In (2) a criterion of optimal transfers is established for a considerably more general problem. There is also indicated there a detailed article (3), prepared for print, devoted to the special case of interest to us. Publication of the latter was delayed until 1949 because of the war. Work (3), as, incidentally, also works (1, 4, 6, 10), has remained unknown abroad to this day. There, after Hitchcock' s work (11), in which the problem considered here was formulated, there appeared many works devoted to this problem. Among them one should first of all mention work (12), in which the author, by means of the simplex method developed by him, obtains a

satisfactory solution. In a number of subsequent works this solution is improved; we note work (13), in which a solution is obtained that essentially coincides with that contained in (3).

corresponding to all possible transportation plans from the i-th production point fill, obviously, a simplex M_i with vertices corresponding to the cases when the entire product from the i-th production point is transported to one consumption point. The points a corresponding to all possible total transportation plans fill the polyhedron

$$M = \sum_i M_i.$$

The plans under which the necessary amount of product is delivered to each production point correspond to the points of the axis $B = \{b = c + \lambda d\}$, where $c = (r_1, r_2, \dots, r_n, 0)$, $d = (0, 0, \dots, 0, -1)$. At the same time the transportation costs amount to $-\lambda$. Therefore the optimal plans will be precisely those plans which correspond to the extreme point of intersection of the axis B with the polyhedron M.

Thus the question has been reduced to the problem of an extreme point. In this case one always has $B \cap M \neq \Lambda$. The first n direction coefficients of the resolving hyperplane H_0 (with the $(n+1)$-st equal to 1) are the potentials of the consumption points, and the constant terms in the equations of the hyperplanes parallel to H_0 and supporting the simplexes M_i are equal to the potentials of the corresponding production points, multiplied by the quantity of product produced there.

Remark on the problem of the extreme point of intersection of an axis with an unbounded polyhedron. The investigation of certain questions may lead to a problem on an extreme point in which one or more M_i are unbounded polyhedra, i.e.

$$M_i = \left\{ a_i = \sum_s \mu_s a_{is} + \sum_t \nu_t a'_{it} + \sum_r \rho_r a''_{ir} \right\},$$

where $\mu_s \geq 0$; $\sum_s \mu_s = 1$; $\nu_t \geq 0$; ρ_r are arbitrary real numbers. To solve such a problem it is enough to solve the problem of the intersection of the axis with the bounded polyhedron

$$\overline{M} = \sum_{i=1}^{m+1} \overline{M}_i,$$

in which $\overline{A}_i = \{a_{is}\}$ $(i = 1, 2, \dots, m)$, $\overline{A}_{m+1} = \{0, Na'_{it}, \pm Na''_{ir}\}$, where N is an undetermined large number. In this case, however, it may turn out that

$B \cap \overline{M} \neq \Lambda$, but $\overline{\lambda}_0$ depends on N. Then the polyhedron M contains an infinite part of the axis B, and

$$\sup_{c+\lambda d \in M} \lambda = +\infty.$$

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