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Abstract

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IRREDUCIBLE UNITARY REPRESENTATIONS OF THE GROUP OF THIRD-ORDER MATRICES PRESERVING AN INDEFINITE HERMITIAN FORM

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1. In our note ([1]) certain series of irreducible unitary representations of the group of unimodular matrices of order n preserving an indefinite Hermitian form were described. In the present note, for the case of the group of matrices of order 3, a complete description is given of all irreducible unitary representations occurring in the decomposition of the regular representation.*

We consider the group G of unimodular matrices of order 3 preserving the Hermitian form

$$H(z, \bar{z}) = -z_1\bar{z}_1 - z_2\bar{z}_2 + z_3\bar{z}_3.$$

The most general approach to the description of all irreducible unitary representations of the group G is as follows.

Denote by M the manifold of points $\zeta = (\zeta_1, \zeta_2)$ of the two-dimensional complex space for which $|\zeta_1|^2 + |\zeta_2|^2 = 1$. Let $\mathcal{H}$ be the space of continuous functions on M. To each matrix $g = \|g_{ij}\|$ from G we associate a transformation of the manifold M: $\zeta \to \zeta_g = (\zeta_1', \zeta_2')$, where

$$\zeta_1' = \frac{\zeta_1 g_{11} + \zeta_2 g_{21} + g_{31}}{\zeta_1 g_{13} + \zeta_2 g_{23} + g_{33}}; \qquad \zeta_2' = \frac{\zeta_1 g_{12} + \zeta_2 g_{22} + g_{32}}{\zeta_1 g_{13} + \zeta_2 g_{23} + g_{33}}.$$

The operators of the representation T_g may initially be defined in the space $\mathcal{H}$ by the formula

$$T_g f(\zeta) = f(\zeta_{\bar{g}})(\zeta_1 g_{13} + \zeta_2 g_{23} + g_{33})^{-\sigma}\overline{(\zeta_1 g_{13} + \zeta_2 g_{23} + g_{33})}^{-\tau}. \qquad (1)$$

Here σ and τ are complex numbers characterizing the representation. From considerations of single-valuedness of expression (1), the difference $\sigma - \tau$ is assumed to be an integer.

If σ and τ are not real integers, then the space $\mathcal{H}$ has no proper invariant subspaces. In this case, in $\mathcal{H}$ one can define, uniquely up to a constant factor, a bilinear functional $A(f_1, \bar{f}_2)$ that commutes with the operators T_g:

$$A(T_g f_1, T_g f_2) = A(f_1, f_2).$$

For certain values of σ, τ this functional turns out to be positive definite. Then, defining the scalar product in $\mathscr{H}$ by means of the bilinear functional A and completing $\mathscr{H}$, we obtain a Hilbert space on which an irreducible unitary representation of the group G is realized. In this way all nondiscrete (principal and supplementary) series of irreducible unitary representations of the group G are constructed.

If σ and τ are real integers, then the space $\mathscr{H}$ always contains proper subspaces invariant with respect to the operators T_g. In this case, in $\mathscr{H}$ one can single out such irreducible

* The irreducible unitary representations of the group of matrices of order 2 were studied in ([2]).

invariant subspaces on which the bilinear functional, permutable with the operators T_g, is sign-definite. In this way one can obtain all the discrete series of irreducible unitary representations of the group G. However, the direct determination of such subspaces of the space $\mathscr{H}$ presents certain difficulties. Below we give another approach to the description of the discrete series of unitary representations of the group G.

2. We shall regard G as a group of linear transformations of the three-dimensional complex vector space of vectors $z = (z_1, z_2, z_3)$; the vector z' into which the vector z is transformed by the matrix g will be denoted by zg.

A function $f(z)$, given in the domain where

$$-|z_1|^2 - |z_2|^2 + |z_3|^2 \geq 0,$$

or in the domain where

$$-|z_1|^2 - |z_2|^2 + |z_3|^2 \leq 0,$$

will be called harmonic if it satisfies the equation

$$\left(-\frac{\partial^2}{\partial z_1 \partial \bar{z}_1} - \frac{\partial^2}{\partial z_2 \partial \bar{z}_2} + \frac{\partial^2}{\partial z_3 \partial \bar{z}_3}\right) f = 0. \tag{2}$$

We shall say that $f(z)$ is a homogeneous function of type (σ, τ) if $f(z)$ satisfies the following homogeneity condition:

$$f(tz) = t^{-\bar{\sigma}} t^{-\tau} f(z). \tag{3}$$

For reasons of single-valuedness it is always assumed here that $\sigma - \tau$ is an integer.

Let $H^+_{\sigma,\tau}$ be the set of all harmonic functions $f(z)$ of type (σ,τ), defined in the domain where

$$-|z_1|^2 - |z_2|^2 + |z_3|^2 \geq 0,$$

for which

$$\|f\|^2 = \int_{H(z,z)=c} |f(z)|^2 \, d\mu(z). \tag{4}$$

Here the integration is carried out over the hyperboloid

$$-|z_1|^2 - |z_2|^2 + |z_3|^2 = c,$$

where $c > 0$, and $d\mu(z)$ denotes the measure on the hyperboloid invariant with respect to transformations of the group G.

Completing $H^+_{\sigma,\tau}$ in the norm (4) and introducing a scalar product in the corresponding way, we obtain a Hilbert space denoted by $\bar{H}^+_{\sigma,\tau}$. In an analogous manner, considering functions in the domain where

$$-|z_1|^2 - |z_2|^2 + |z_3|^2 \leq 0,$$

we define the spaces $H^-_{\sigma,\tau}$ and $\bar{H}^-_{\sigma,\tau}$.

Putting

$$w_1 = \frac{z_1}{z_3}, \qquad w_2 = \frac{z_2}{z_3},$$

we represent a function $f(z)$ of type (σ,τ) in the form

$$f(z) = z_3^{-\bar{\sigma}} \bar{z}_3^{-\tau} \varphi(w_1, w_2). \tag{5}$$

Therefore, instead of functions of type (σ,τ), one may consider functions $\varphi(w) = \varphi(w_1, w_2)$, defined in the domain

$$|w_1|^2 + |w_2|^2 \leq 1$$

or in the domain

$$|w_1|^2 + |w_2|^2 \geq 1$$

of the two-dimensional complex projective space. The norm in $H^\pm_{\sigma,\tau}$, in terms of the functions $\varphi(w)$, takes the form

$$\|\varphi\|^2 = \int |\varphi(w)|^2 \left|1 - |w_1|^2 - |w_2|^2\right|^{\mathrm{Re}(\sigma+\tau)-3} \, dw_1 dw_2, \tag{6}$$

where the integration is carried out respectively over the domain

$$|w_1|^2 + |w_2|^2 \leq 1$$

or over the domain

$$|w_1|^2 + |w_2|^2 \geq 1.$$

Substituting (5) into equation (2), we obtain the equation for the functions $\varphi(w)$:

$$\Big\{(1-|w_1|^2)\frac{\partial^2}{\partial w_1 \partial \bar{w}_1} + (1-|w_2|^2)\frac{\partial^2}{\partial w_2 \partial \bar{w}_2} - w_1\bar{w}_2\frac{\partial^2}{\partial w_1 \partial \bar{w}_2} - \bar{w}_1 w_2\frac{\partial^2}{\partial \bar{w}_1 \partial w_2}$$
$$-\tau\left(w_1\frac{\partial}{\partial w_1} + w_2\frac{\partial}{\partial w_2}\right) - \sigma\left(\bar{w}_1\frac{\partial}{\partial \bar{w}_1} + \bar{w}_2\frac{\partial}{\partial \bar{w}_2}\right) + \sigma\tau\Big\}\varphi = 0^*. \qquad (7)$$

* When $\sigma = \tau = 0,$ this equation defines functions studied in detail for the case of the domain $|w_1|^2+|w_2|^2 <$

For reasons of convergence of the integral (6), we shall henceforth assume that $\mathrm{Re}(\sigma+\tau) > 2$.

Theorem 1. If σ and τ are not integers, then $H^+_{\sigma,\tau} = 0$, $H^-_{\sigma,\tau} = 0$. If σ and τ are integers, then $H^-_{\sigma,\tau} \neq 0$, $H^+_{\sigma,\tau} \neq 0$ if and only if $\sigma \leq 0$ or $\tau \leq 0$.

Theorem 2. The functions $\varphi(w) \in H^+_{\sigma,\tau}$ and $\varphi(w) \in H^-_{\sigma,\tau}$ are uniquely determined by their boundary values on the manifold $M: \ |w_1|^2 + |w_2|^2 = 1$.

In the space $\bar{H}^\pm_{\sigma,\tau} \neq 0$ one can define a unitary representation T_g of the group G by the formula

$$T_g f(z) = f(zg). \qquad (8)$$

On passing to the functions $\varphi(w)$, formula (8) for the operator T_g takes the form

$$T_g\varphi(w) = \varphi(w\bar{g})\,(w_1 g_{13} + w_2 g_{23} + g_{33})^{-\sigma}(\bar{w}_1 g_{13} + \bar{w}_2 g_{23} + g_{33})^{-\tau}. \qquad (8')$$

Theorem 3. The representations of the group G given in the nonzero spaces $\bar{H}^+_{\sigma,\tau}$ and $\bar{H}^-_{\sigma,\tau}$ by formula (8) are unitary and irreducible.

By virtue of Theorem 2, these representations may also be given in the space of functions on the manifold M that are boundary values of functions from $\bar{H}^+_{\sigma,\tau}$ (or from $\bar{H}^-_{\sigma,\tau}$).

3. We now indicate how the representations described above decompose into irreducible representations of the maximal compact subgroup $\mathfrak{U}$ of the group G. The subgroup $\mathfrak{U}$ of the group G is isomorphic to the full group of unitary matrices of order 2. Therefore the irreducible representations of the subgroup $\mathfrak{U}$ are characterized by a pair of integers $f_1 \geq f_2$. The space $H_{(f_1,f_2)}$ in which the corresponding representation acts has dimension $f_1 - f_2 + 1$.

Theorem 4. Each of the nonzero spaces $\bar{H}^\pm_{\sigma,\tau}$ decomposes, with respect to the representations of the subgroup $\mathfrak{U}$, into a direct sum of pairwise inequivalent subspaces $H_{(f_1,f_2)}$. The indices f_1, f_2 of the subspaces entering into the decomposition of $\bar{H}^\pm_{\sigma,\tau}$ are given by the expressions

$$f_1 = 2p - q + (\sigma - \tau), \qquad f_2 = p - 2q + (\sigma - \tau). \tag{9}$$

Here, if $\sigma > 0,\ \tau > 0$, then $p = \tau-1, \tau, \tau+1, \ldots;\ q = \sigma-1, \sigma, \sigma+1, \ldots$.* If $\sigma \le 0$, then for $H^+_{\sigma,\tau}$, $p = 0, 1, \ldots, -\sigma$, while q runs through all nonnegative integers; for $H^-_{\sigma,\tau}$, $p = \tau - 1, \tau, \tau + 1, \ldots$, while q runs through all nonnegative integers. An analogous description is obtained also for $\tau \le 0$.

The highest-weight vectors φ_{f_1,f_2} from $H_{(f_1,f_2)}$ have the following form:

For the space $\bar{H}^-_{\sigma,\tau} \ne 0$,

$$\varphi_{f_1,f_2} = w_1^p \bar{w}_2^q \left(|w_1|^2 + |w_2|^2\right)^{-(p+q+1)} \times$$

$$\times F\left(\tau - 1 - p, \sigma - 1 - q;\ \sigma + \tau - 1;\ 1 - |w_1|^2 - |w_2|^2\right).$$

Here F is the hypergeometric function, which in the present case is a polynomial in $1 - |w_1|^2 - |w_2|^2$ of degree $\min(p - \tau + 1, q - \sigma + 1)$. The indices p, q are related to f_1, f_2 by relation (9).

For the space $H^+_{\sigma,\tau}$, where $\sigma \le 0$,

$$\varphi_{f_1,f_2} = w_1^p \bar{w}_2^q G_{-\sigma-p}\left(p + q + \sigma + \tau,\ p + q + 2;\ |w_1|^2 + |w_2|^2\right),$$

where $G_{-\sigma-p}$ is a Jacobi polynomial.

* Recall that in this case the space $\bar{H}^-_{\sigma,\tau}$ is meant, since $\bar{H}^+_{\sigma,\tau} = 0$.

For the space $\bar{H}^+_{\sigma,\tau}$, where $\tau \leqslant 0$:

$$\varphi_{f_1,f_2} = w_1^p \bar{w}_2^q G_{-\tau-q}(p + q + \sigma + \tau,\ p + q + 2;\ |w_1|^2 + |w_2|^2).$$

The representations realized in the spaces $\bar{H}^+_{\sigma,\tau} \ne 0$ are pairwise inequivalent. They all contain an irreducible representation of the subgroup U of dimension 1. Another description of these representations was given in ([1]). The representations realized in $\bar{H}^-_{\sigma,\tau}$ for $\sigma > 0$ and $\tau > 0$ are likewise pairwise inequivalent. On the other hand, for $\sigma \leqslant 0$, the representations realized in the spaces $\bar{H}^-_{\sigma,\tau}$ and $\bar{H}^-_{2-\sigma,\sigma+\tau-1}$ are equivalent to one another (an analogous assertion holds for $\tau \leqslant 0$). All representations realized in the spaces $\bar{H}^-_{\sigma,\tau}$ contain a unique representation of the subgroup U of lowest dimension, and this dimension is always greater than one.

Thus, three discrete series of irreducible unitary representations of the group G have been obtained: 1) representations realized in the spaces $\bar{H}^+_{\sigma,\tau}$, where $\sigma \leqslant 0$;

2) representations realized in the spaces $\bar{H}^{+}_{\sigma,\tau}$, where $\tau \leqslant 0$; 3) representations realized in the spaces $\bar{H}^{-}_{\sigma,\tau}$, where $\sigma > 0,\ \tau > 0$.

4. In the preceding arguments it was assumed throughout that $\mathrm{Re}(\sigma+\tau) > 2$. However, one may also consider the limiting case when $\mathrm{Re}(\sigma+\tau) = 2$, if $\|\varphi\|^2$ is defined by the formula

$$\|\varphi\|^2 = \left\{ \lim_{\lambda\to 2+0} (\lambda-2) \int |\varphi(w)|^2 \left|1 - |w_1|^2 - |w_2|^2\right|^{\lambda-3} dw_1 dw_2. \qquad (10)$$

The integral (10) then reduces to an integral over the boundary $|\zeta_1|^2 + |\zeta_2|^2 = 1$

$$\|\varphi\|^2 = \int |\varphi(\zeta)|^2 \, d\mu(\zeta),$$

where $d\mu(\zeta) = c\, d|\zeta_1|^2\, d(\arg\zeta_1)\, d(\arg\zeta_2)$. In view of this, it is natural to define the representation in the space of functions given on the boundary. As a result we obtain the principal non-discrete series of irreducible unitary representations of the group G, specified by the integer $n = \sigma - \tau$ and the real number $\rho = \mathrm{Im}(\sigma+\tau)$, which was described earlier in ([1]).

Theorem 5. *The regular representation of the group G decomposes into the representations of the discrete series described above and of the principal non-discrete series.*

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[1] M. I. Graev, DAN, **98**, No. 4 (1954). [2] V. Bargmann, Ann. Math., **48**, 3 (1947). [3] Hua Lo-keng, Acta Math. Sinica, **2** (1952); **5** (1955); **6** (1956) (Chinese) (the materials of these articles are also presented in a brochure published in English: *Harmonic Analysis of the Classical Domain in the Study of Analytic Functions of Several Complex Variables*).

Note: Figure translations are in progress. See original paper for figures.