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Abstract

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THE GROWTH OF THE LENGTH OF LARGE WIND WAVES AND THE ROLE OF INTERNAL TURBULENT FRICTION

In one of our papers (1), the law of growth of the length of wind waves, previously found by us in the first approximation, was refined on the basis of the theorem on the moment of momentum and on the basis of corrected expressions for the kinetic and potential energy of waves.

The obtained law of variation of the steepness of growing waves

$$h/\lambda = 0.04 + 0.103(\lambda_0/\lambda)^{3/2} \tag{1}$$

proved to be valid not only at intermediate stages, where it was compared with the results of experiments in a storm basin, but even at the limit—when the wavelengths tend toward the maximum possible values. In full agreement with observations in the ocean, the ratio of wave height h to their length λ tends to the limiting value 1/25.

The success of the theoretical derivation is explained by the fact that the growth of the moment of momentum was analytically connected with the derivative of the total energy actually possessed by the waves at the corresponding stages of development. Consequently, the dissipation of a certain amount of the energy received from the wind into turbulent friction in the sea water during wave motion was automatically taken into account. The period-averaged moment of the external forces $\overline{M}$, which appeared in the derivations, automatically included all forces acting on the water particles.

In the present article we attempt to introduce a further refinement into the study of this question, in order to reveal a kind of second-order effect caused by internal friction: we shall consider separately the dissipation of energy and the dissipation of momentum due to turbulent exchange in water. On the basis of our paper (2), we write the expressions

$$|W_\mu| = \frac{4\nu}{R^2} E; \tag{2}$$

$$\frac{d\overline{Q}}{dt} = \overline{M} - \frac{4\nu_1}{R^2}\,\overline{Q}. \tag{3}$$

Here W_μ is the power expended on friction; E is the total energy of the waves per unit area of the sea surface; ν is the coefficient of turbulent exchange (kinematic viscosity); R is the radius of the rolling circle ($R = \lambda/2\pi$); $\overline{Q}$ is the averaged moment of momentum of the water masses, also calculated per unit area of the sea surface; $\overline{M}$ is the averaged moment of the external forces acting on the wave (from the wind); ν_1 is a coefficient analogous to ν, in all probability equal to ν. (In the present work we shall attempt to equate ν_1 and ν, and on the basis of the results judge the validity of such an assumption.)

The total power W_v transmitted to the wave by the wind consists of the increase in wave energy per unit time, manifested in the growth

wave heights, and from compensating the power lost to internal turbulent friction. Thus, on the basis of (2),

$$W_\nu = \frac{dE}{dt} + \frac{4\nu}{R^2}E. \tag{4}$$

Our previous investigations ([3]) make it possible to determine the averaged moment of the acting forces, knowing W_ν and the angular velocity ω of the orbital motion of water particles in the wave:

$$\overline{M} = \frac{W_\nu}{\omega} = \frac{1}{\omega}\frac{dE}{dt} + \frac{4\nu}{R^2}\frac{E}{\omega}. \tag{5}$$

The wave energy E, referred to a unit surface area of the sea, must be taken with A. I. Nekrasov' s correction ([4]) in the kinetic part and with our correction in the potential part. As a result, it is expressed ([1]) as follows:

$$E = \frac{1}{2}\delta g r^2(1 + 2r/R), \qquad \text{where } r = h/2. \tag{6}$$

On the basis of (6) we express the derivative dE/dt in (5). The expression for the derivative $d\overline{Q}/dt$ we borrow from our earlier investigations ([5]). Then the generalized expression (3) of the theorem on the moment of external forces and the moment of momentum of water masses (averaged over one wave period T) is written in the form

$$\delta cr\frac{dr}{dt} + 3\delta g\frac{r^2}{c}\frac{dr}{dt} - \delta\frac{g}{c}\frac{r^3}{R}\frac{dR}{dt} + \frac{4\nu}{R^2}\frac{E}{\omega} = \delta cr\frac{dr}{dt} + \frac{1}{2}\delta r^2\frac{dc}{dt} + \frac{4\nu}{R^2}\overline{Q}. \tag{7}$$

We transfer to the right-hand side the last (additional) term from the left-hand side. The generalized expression obtained differs from the expression derived in work ([1]) only by the quantity

$$\frac{4\nu}{R^2}\left(\overline{Q}-\frac{E}{\omega}\right).$$

With the aid of the expression for the energy (6), the expression for the averaged moment of momentum $\overline{Q}$ (from our earlier works), and the elementary relations between wave characteristics, we find

$$\frac{4\nu}{R^2}\left(\overline{Q}-\frac{E}{\omega}\right)=\frac{4\nu}{R^2}\frac{1}{2}\delta r^2 c\left[1-1-2\frac{r}{R}\right]=-\frac{2\nu}{R^3}\delta r^3 c. \tag{8}$$

Taking (8) into account, expression (7) is easily simplified after multiplying both its sides by $c/\delta r^2$ (analogously to the derivation of equation (14) in work ([1])). As a result of simple transformations, after canceling all terms by g, one obtains an equation that differs from equation (15) of work ([1]) only by an additional term with coefficient ν:

$$\left(\frac{1}{4}+\frac{r}{R}\right)\frac{dR}{dt}=3\frac{dr}{dt}+2\frac{r}{R}\frac{\nu}{R}. \tag{9}$$

As we see, in contrast to the simple conditions of the preceding work, the growth of R (and therefore of the wavelength λ) is now connected not only with the rate of growth of r (and therefore with the rate of increase of wave height h), but also with a new term depending on internal turbulent friction. It is evident that, on the one hand, the role of this additional term is obviously small when waves rapidly grow in height at the initial stages and, at the same time, the numerical value of ν is small; on the other hand, the role of the additional term should appear most distinctly upon attainment of the maximum wave height, when dr/dt becomes equal to zero.

As applied to this case, we rewrite equation (9), without the first term on the right-hand side, as follows:

$$\left(\frac{1}{4}+\frac{r}{R}\right)R\frac{dR}{dt}=2\frac{r}{R}\nu. \tag{10}$$

On the basis of present-day measurements of waves in the ocean, one may, with sufficient approximation, regard the quantity r/R as varying only weakly over finite

stages. We shall take it, in a first approximation, to be constant, and then check the validity of such an assumption. Then, rewriting (10) somewhat differently,

$$\frac{d\left(R^2\right)}{dt}=\frac{16\nu}{R/r+4}, \tag{11}$$

Fig. 1

Figure 1: Fig. 1

one may assume that the quantities on the right-hand side of (11) are practically constant, and the integral of (11) will be written in the simple form:

$$R^2 = \frac{16\nu}{R/r+4}\,t + R_1^2. \tag{12}$$

Here the integration constant R_1^2 is the value of the square of the radius of the rolling circle that corresponds to the attained maximum wave height. As we see, the radius of the rolling circle R, and hence the wavelength λ, must increase according to the parabolic law (12), which may also be expressed as

$$\frac{R}{R_1} = \left(1 + \frac{16}{R/r+4}\frac{\nu t}{R_1^2}\right)^{1/2}. \tag{13}$$

Let us see how appreciable the increase in wavelength is under the influence of this effect after the wave height has reached its maximum. Let us set, for orientation: $h/\lambda = 1/25$, whence $r/R = 2\pi h/\lambda = 0.25$; $R/r = 4$; $\lambda = 270$ m, and therefore $R = 4.3 \cdot 10^3$ cm; $\nu = 100$ cm^2 sec^{-1}; $t = 1$ hour $= 3.6 \cdot 10^3$ sec. Then, on the basis of (13), it follows that approximately $R/R_1 = 1.02$. In other words, under the influence of the effect under consideration, over the course of 1 hour the radius of the rolling circle, and consequently the wavelength, increases by only 2% of the value corresponding to the maximum wave height.

Fig. 1

The increase in wavelength is inseparably connected with an increase in their phase velocity c, and an increase in c inevitably leads to a decrease in the relative wind velocity $(V-c)$ at the same value of the absolute wind velocity V. On the other hand, the power W_V transmitted by the wind to the wave is proportional to $(V-c)^2$. Consequently, the effect under investigation must lead to a violation of the equality between the power transmitted by the wind and the power absorbed by internal turbulent friction. As a result, the wave height must decrease after it has passed through a maximum. But the decrease in wave height with time changes the sign of the derivative dr/dt in equation (9). Thus the phenomenon under investigation is either weakened still more, or is completely eliminated, if at the corresponding particular values it turns out that (for $dr/dt < 0$)

$$3\frac{dr}{dt} + 2\frac{r}{R}\frac{\nu}{R} = 0.$$

Let us recall that an increase in wavelength after their height reaches a maximum was indeed observed in experiments in a storm basin ([6]).

In Fig. 1 the results of one of the series of experiments are reproduced. The diagram shows that the wave height passes through a maximum. It decreases precisely as a consequence of the increase in wavelength, the increase in their phase velocity c, and the simultaneous decrease in the relative wind velocity $(V-c)$.

Under shallow-water conditions the limiting wave height was proportional to the first power of $(V-c)$, and nevertheless the wave height, after passing through a maximum, decreased by 10% in 10 min. Under ocean or deep-sea conditions the limiting

the height of the waves is proportional to $(V-c)^2$. Hence, even with a much slower increase in wavelength and a decrease in the relative wind speed, a noticeable decrease in wave height must occur in the ocean. In all probability, this ensures that the right-hand side of equation (9) becomes equal to zero and leads to the emergence of a stable length of wind waves, somewhat greater than that at which the wave height has reached its maximum. It must be supposed that it is precisely for this reason that wind waves are sometimes observed in the ocean characterized by the ratio $c/V = 0.80$, whereas the maximum heights [(2)] of wind waves correspond to $c/V = 0.75$.

The results obtained confirm the correctness of the assumptions made. First of all, there is no doubt as to the possibility of considering the quantity r/R approximately constant in carrying out the integration (11). Then, apparently, in reality ν, which characterizes the exchange of energy between the layers of agitated water, is practically equal to the quantity ν_1, which characterizes the exchange of momentum. Finally, the basic assumption made at the beginning of the present article also proves to be valid: owing to the scattering of momentum simultaneously with the scattering of wave energy under the action of turbulent exchange between layers, the equations of the generalized problem lead to the same consequences as followed from the elementary conditions of the preceding work [(1)]. Consequently, equations (21), (21′), and (22) in work [(1)] must practically remain valid at all stages of the development of wind waves in the ocean, up to the appearance of extremely high waves corresponding to the given wind speed V. Thereafter the new phenomenon that we have just described sets in.

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CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.