MATHEMATICS

1957

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Abstract

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MATHEMATICS

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ON BEST APPROXIMATIONS OF ALMOST-PERIODIC FUNCTIONS BY ENTIRE FUNCTIONS OF FINITE DEGREE

(Presented by Academician V. I. Smirnov, 20 V 1957)

1. Denote by B_λ the class of entire functions of degree $\leqslant \lambda$, bounded on the real axis. Let the function $f(z)$ be defined and bounded on the real axis. Put

$$E_\lambda(f) = \inf_{F(z)\in B_\lambda} \left\{ \sup_x |f(x) - F(x)| \right\}.$$

We shall say that an almost-periodic function $f(x)$ belongs to the class Π if the Fourier series of this function has the form

$$\sum_{k=-\infty}^{\infty} A_k e^{i\lambda_k x} \quad (\lambda_0 = 0;\ \lambda_k > 0,\ \lambda_{k+1}/\lambda_k = q_k \geqslant \theta > 1 \text{ for } k > 0;\ \lambda_k = -\lambda_{-k}).$$

Put

$$R_\lambda(f) = \sup_x \left| f(x) - \sum_{|\lambda_k|\leqslant\lambda} A_k e^{i\lambda_k x} \right|, \qquad \alpha_\lambda(f) = \sum_{|\lambda_k|>\lambda} |A_k|.$$

2. Theorem 1. If $f(x) \in \Pi$, then $R_\lambda(f) \leqslant C(\theta) E_\lambda(f)$, where $C(\theta)$ is a constant depending only on θ.

The proof of the theorem is based on the following two lemmas.

Lemma 1. If $F(z) \in B_\lambda$,

$$\Psi_{a,b}(u) = \frac{2}{\pi(b-a)} \frac{\sin\frac{b+a}{2}u \, \sin\frac{b-a}{2}u}{u^2},$$

where $\lambda < a < b$, then

$$F_{a,b}(x) = \int_{-\infty}^{\infty} F(x+u)\Psi_{a,b}(u)\,du = F(x).$$

Proof. By virtue of the inequality ((1), p. 76)

$$\int_{-\infty}^{\infty} |\Psi_{a,b}(u)|\,du \leqslant \frac{4}{\pi} + \frac{2}{\pi}\ln\frac{b+a}{b-a} \tag{1}$$

$F_{a,b}(x)$ exists for any function $F(z) \in B_{\lambda}$.

On the basis of the Wiener-Paley theorem ((2), p. 151)

$$F(x) = F(0) + \frac{x}{\sqrt{2\pi}}\int_{-\lambda}^{\lambda} e^{itx}\varphi(t)\,dt, \tag{2}$$

where $\varphi(t) \in L_2(-\lambda,\lambda)$; therefore

$$F_{a,b}(x) = F(0)\int_{-\infty}^{\infty}\Psi_{a,b}(u)\,du + \frac{x}{\sqrt{2\pi}}I_1(x) + \frac{1}{\sqrt{2\pi}}I_2(x),$$

where

$$I_1(x) = \int_{-\lambda}^{\lambda}\varphi(t)\int_{-\infty}^{\infty} e^{it(x+u)}\Psi_{a,b}(u)\,du\,dt,$$

$$I_2(x) = \int_{-\lambda}^{\lambda}\varphi(t)e^{itx}\int_{-\infty}^{\infty} e^{itu}u\Psi_{a,b}\times(u)\,du\,dt.$$

Taking into account that

$$\int_{-\infty}^{\infty}\Psi_{a,b}(u)\,du = 1, \qquad \int_{-\infty}^{\infty} e^{it(x+u)}\Psi_{a,b}(u)\,du =$$

$$= e^{itx} \quad \text{for } |t| < a \quad ((1),\ \text{pp. } 76, 77)$$

and

$$\int_{-\infty}^{\infty} e^{itu}u\Psi_{a,b}(u)\,du = 0,$$

we obtain

$$F_{a,b}(x) = F(x).$$

Lemma 2. If $F(z) \in B_\lambda,\ \lambda < |\Lambda|$, then

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T F(x) e^{-i\Lambda x}\, dx = 0.$$

Proof follows from (2) and the Riemann–Lebesgue theorem ((3), p. 19).

Proof of Theorem 1 (6). If $F(z) \in B_{\lambda_k}$ and $\lambda_k' = \dfrac{\lambda_k + \lambda_{k+1}}{2}$, then, by Lemma 1,

$$\int_{-\infty}^{\infty} F(x+u) \Psi_{\lambda_k', \lambda_{k+1}}(u)\, du = F(x). \tag{3}$$

If $f(x) \in L$, then ((1), p. 76)

$$\int_{-\infty}^{\infty} f(x+u) \Psi_{\lambda_k', \lambda_{k+1}}(u)\, du = \sum_{|\lambda_\nu| \le \lambda_k} A_\nu e^{i\lambda_\nu x}. \tag{4}$$

In the class B_{λ_k} there exists a function $\widetilde{F}(z)$ such that ((4), p. 371)

$$|f(x) - \widetilde{F}(x)| \le E_{\lambda_k}(f). \tag{5}$$

From inequalities (5) and (1) it follows that

$$\left| \int_{-\infty}^{\infty} \widetilde{F}(x+u) \Psi_{\lambda_k', \lambda_{k+1}}(u)\, du - \int_{-\infty}^{\infty} f(x+u) \Psi_{\lambda_k', \lambda_{k+1}}(u)\, du \right| \le$$

$$\le E_{\lambda_k}(f) \left(\frac{4}{\pi} + \frac{2}{\pi} \ln \frac{3\theta + 1}{\theta - 1} \right). \tag{6}$$

From (5) and (6), taking into account (3) and (4), we obtain

$$R_{\lambda_k'}(f) \le \left[1 + \frac{2}{\pi} \left(2 + \ln \frac{3\theta+1}{\theta-1} \right) \right] E_{\lambda_k}(f). \tag{7}$$

Let $\lambda_{k-1} < \lambda < \lambda_k$; then

$$R_\lambda(f) \le |A_k| + |A_{-k}| + R_{\lambda_k}(f). \tag{8}$$

By Lemma 2,

$$A_k = \lim_{T\to\infty} \frac{1}{T} \int_0^T [f(x) - F(x)] e^{-i\lambda_k x}\, dx,$$

where $F(x)$—the···

an arbitrary function belonging to the class B_λ; hence $|A_k| \le E_\lambda(f)$, $|A_{-k}| \le E_\lambda(f)$, and from inequalities (8) and (7) it follows that for any λ

$$R_\lambda(f) \le C(\theta) E_\lambda(f), \quad \text{where} \quad C(\theta) = 3 + \frac{2}{\pi}\left(2 + \ln\frac{3\theta+1}{\theta-1}\right).$$

Corollary. If $f(x) \in \mathscr{L}$, then the order equalities $E_\lambda(f) \sim R_\lambda(f) \sim \alpha_\lambda(f)$ hold.

Proof. From Theorem 1 of paper (^{8}) there follows the inequality $\alpha_\lambda(f) \le C_1(\theta) R_\lambda(f)$, where $C_1(\theta)$ is a constant depending only on θ.

The corollary of Theorem 1 is a generalization of Theorem 3 of paper (^{8}).

Theorem 2. Let the function $f(z)$ be defined and bounded on the real axis. If there exists a sequence of nonnegative (nonpositive) numbers $\{x_l\}$ $\left(l = 0, 1, 2, \dots;\ |x_l| < |x_{l+1}|;\ \lim_{l\to\infty} \frac{l}{|x_l|} > \frac{\lambda}{\pi}\right)$ such that for some function $F_0(z) \in B_\lambda$

$$\operatorname{Re}\{f(x_l) - F_0(x_l)\} = (-1)^l L_l, \tag{9}$$

where $L_l \ge L > 0$, then $E_\lambda(f) \ge L$.

Proof. Suppose that there exists a function $\Phi(z) \in B_\lambda$ for which

$$(-1)^l \Phi(x_l) \ge 1 \quad (l = 0, 1, 2, \dots). \tag{10}$$

Let $n(t)$ be the number of zeros of $\Phi(z)$ in the domain $|\arg z| < \varepsilon,\ |z| < t$, if the terms of the sequence $\{x_l\}$ are nonnegative, and let $n(t)$ be the number of zeros of $\Phi(z)$ in the domain $|\pi - \arg z| < \varepsilon,\ |z| < t$, if the terms of the sequence $\{x_l\}$ are nonpositive. It is known (^{5}), Ch. V, § 4, Theorem 11, that for arbitrarily small ε

$$\lim_{t\to\infty} \frac{n(t)}{t} = \frac{d}{2\pi},$$

where d is the length of the indicator diagram of the function $\Phi(z)$.

On the basis of Pólya' s theorem ((^{5}), Ch. I, § 20) $d \le 2\lambda$, therefore

$$\lim_{t\to\infty} \frac{n(t)}{t} \le \frac{\lambda}{\pi}. \tag{11}$$

There exists a subsequence $\{x_{l_k}\}$ $(k = 0, 1, 2, ...)$ of the sequence $\{x_l\}$ such that

$$\lim_{k\to\infty} \frac{l_k}{|x_{l_k}|} > \frac{\lambda}{\pi}. \tag{12}$$

Taking (10) and (12) into account, we obtain

$$\lim_{k\to\infty} \frac{n(|x_{l_k}|)}{|x_{l_k}|} \geq \lim_{k\to\infty} \frac{l_k}{|x_{l_k}|} > \frac{\lambda}{\pi},$$

which contradicts (11). Thus the inequalities (10) are impossible at all points x_l; consequently, the set of these points is a set of uniqueness of degree λ ((4), p. 376).

Let

$$F_0(z) = \sum_{k=0}^{\infty} c_k \frac{z^k}{k!};$$

put

$$F_1(z) = \sum_{k=0}^{\infty} \frac{c_k + \overline{c_k}}{2} \frac{z^k}{k!};$$

obviously, $F_1(z) \in B_\lambda$ and $\operatorname{Re} F_0(x) = F_1(x)$. From (9) it follows

$$(-1)^l \{\operatorname{Re} f(x_l) - F_1(x_l)\} \geq L. \tag{13}$$

Consequently, by (13) ((4), p. 376, Theorem IV) $E_\lambda(f) \geq L$.

Theorem 3. If $f(x) \in \mathcal{L}$, $\theta > 3$, and $\arg A_{-k} = -\arg A_k$, then

$$\alpha_\lambda(f) \leq \frac{1}{\cos\dfrac{\pi}{\theta - 1}} E_\lambda(f).$$

Proof ((7), Theorem 5). Let

$$x_p^{(k)} = \frac{(4+p)\pi - \varphi_k}{\lambda_k},$$

where k is a natural number, $p = 0, 1, 2, ...$, and $\varphi_k = \arg A_k$. Considering p and k fixed, denote by $x_{p+2r_1}^{(k+1)}$ the point of the form $x_{p+2r}^{(k+1)}$, where r is an integer, nearest to $x_p^{(k)}$. Obviously,

$$\left| x_p^{(k)} - x_{p+2r_1}^{(k+1)} \right| \leq \pi/\lambda_{k+1}.$$

Denote by $x_{p+2r_2}^{(k+2)}$ the point of the form $x_{p+2r}^{(k+2)}$ nearest to the point $x_{p+2r_1}^{(k+1)}$, and so on.

Thus we obtain a sequence $\{x_{p+2r_l}^{(k+l)}\}$ $(l = 0, 1, 2, \ldots;\ r_0 = 0)$ such that

$$\left|x_{p+2r_l}^{(k+l)} - x_{p+2r_{l+1}}^{(k+l+1)}\right| \leq \pi/\lambda_{k+l+1}.$$

There exists

$$\lim_{l\to\infty} x_{p+2r_l}^{(k+l)} = \tilde{x}_p^{(k)}$$

and the inequality holds

$$\left|x_{p+2r_l}^{(k+l)} - \tilde{x}_p^{(k)}\right| \leq \frac{\pi}{\lambda_{k+1}} \frac{1}{\theta - 1}. \tag{14}$$

Putting $l = 0$ in inequality (14), we obtain the following properties of the numbers $\tilde{x}_p^{(k)}$ $(p = 0, 1, 2, \ldots)$:

$$\tilde{x}_0^{(k)} > 0, \qquad \tilde{x}_p^{(k)} < \tilde{x}_{p+1}^{(k)}, \qquad \lim_{p\to\infty} \frac{p}{\tilde{x}_p^{(k)}} = \frac{\lambda_k}{\pi}. \tag{15}$$

Let $\lambda_{k-1} \leq \lambda < \lambda_k$; then

$$\operatorname{Re}\left\{f(\tilde{x}_p^{(k)}) - \sum_{|\lambda_\nu|<\lambda} A_\nu e^{i\lambda_\nu \tilde{x}_p^{(k)}}\right\} = (-1)^p L_p, \tag{16}$$

where, by virtue of (14),

$$L_p = \sum_{l=0}^{\infty} \{|A_{k+l}| + |A_{-(k+l)}|\} \cos \lambda_{k+l} (\tilde{x}_p^{(k)} - x_{p+2r_l}^{(k+l)}) \geq \cos \frac{\pi}{\theta - 1}\, \alpha_\lambda(f).$$

By Theorem 2, from (15) and (16) it follows that

$$E_\lambda(f) \geq \cos \frac{\pi}{\theta - 1}\, \alpha_\lambda(f).$$

Corollary. If $f(x) \in L$ and $q_k \to \infty$, then the asymptotic equalities hold

$$E_\lambda(f) \simeq R_\lambda(f) \simeq \alpha_\lambda(f).$$

The corollary of Theorem 3 is a generalization of Theorem 4 of paper (8).

I express my deep gratitude to Prof. I. P. Natanson for supervising the work.

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Received
15 V 1957

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Note: Figure translations are in progress. See original paper for figures.

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