THEORY OF ELASTICITY

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Abstract

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A PUNCH RESTING ON AN ELASTIC HALF-SPACE WHOSE MODULUS OF ELASTICITY IS A POWER FUNCTION OF DEPTH

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The question of the distribution of pressure over the base of a punch resting on an elastic half-space has attracted the attention of numerous researchers. The solution of a number of important problems on punches is due to N. I. Muskhelishvili and A. I. Lur' e. This question is examined in especially great detail in the monograph of I. Ya. Shtaerman ([1]), and also in the book of L. A. Galin ([2]). Most works are devoted to the problem of a punch resting on an elastic half-space.

It is of interest to extend the class of models of an elastic foundation. In the present note:

1) the problem is formulated for a punch resting on a foundation with a homogeneous axisymmetric kernel ([3]), i.e., it is assumed that the settlement of a point (x, y) of the foundation from a unit vertical force applied at the point (ξ, η) is a function only of the distance between these points:

$$K(r) = K\left(\sqrt{(x-\xi)^2 + (y-\eta)^2}\right);$$

2) the problem is solved for the case when the modulus of elasticity of the foundation varies with depth according to a power law.

1. As shown, for example, in ([3]), if a load $p(x, y)$ acts on the surface of the foundation (to shorten the exposition we shall assume that p is an even function), then

$$w(x,y) = \int_0^\infty \int_0^\infty c(z_1,z_2)\, a(z_1,z_2) \cos z_1 x \cos z_2 y \, dz_1\, dz_2, \quad (1)$$

$$p(x,y) = \int_0^\infty \int_0^\infty a(z_1,z_2) \cos z_1 x \cos z_2 y \, dz_1\, dz_2, \quad (2)$$

where

$$a(z_1, z_2) = \frac{4}{\pi^2} \int_0^\infty \int_0^\infty p(x, y) \cos z_1 x \cos z_2 y \, dx \, dy, \tag{3}$$

$$c(z_1, z_2) = \int_0^\infty \int_0^\infty K(x, y) \cos z_1 x \cos z_2 y \, dx \, dy. \tag{4}$$

If it is assumed that for points $B(x, y)$ lying within the region F, the settlement $w = \varphi(x, y)$, while for points lying outside this region, $p(x, y) = 0$, with $\varphi(x, y)$ known and $p(x, y)$ an unknown function, then the problem is formulated as follows:

$$\int_0^\infty \int_0^\infty c(z_1, z_2) a(z_1, z_2) \cos z_1 x \cos z_2 y \, dz_1 \, dz_2 = \varphi(x, y), \qquad B(x, y) \in F, \tag{5}$$

$$\int_0^\infty \int_0^\infty a(z_1, z_2) \cos z_1 x \cos z_2 y \, dz_1 \, dz_2 = 0, \qquad B(x, y) \in \overline{F}. \tag{6}$$

These equations are a generalization of the dual equations, an exposition of whose theory can be found in ([4,5]). Solving them with respect to a, one can then easily find p.

In the axisymmetric case, using the well-known formula

$$\int_{-\infty}^\infty \int_{-\infty}^\infty p\left(\sqrt{x^2 + y^2}\right) \cos z_1 x \cos z_2 y \, dx \, dy = 2\pi \int_0^\infty r p(r) J_0(\gamma r) \, dr = 2\pi \tilde{p}(\gamma), \tag{7}$$

where $\gamma = \sqrt{z_1^2 + z_2^2}$, $r = \sqrt{x^2 + y^2}$, we obtain

$$a(\gamma) = \frac{1}{2\pi} \tilde{p}(\gamma),$$

and the problem of a punch of radius R, having settlement $w_0(r)$, will be formulated as follows:

$$\text{for } r < R \qquad \int_0^\infty c(\gamma) a(\gamma) \gamma J_0(\gamma r) \, d\gamma = \frac{2}{\pi} w_0, \tag{8}$$

$$\text{for } r > R \qquad \int_0^\infty a(\gamma) \gamma J_0(\gamma r) \, d\gamma = 0, \tag{9}$$

where

$$c(\gamma) = 2\pi \int_0^\infty rK(r)J_0(\gamma r)\,dr. \tag{10}$$

If we denote $\dfrac{r}{R} = \rho$; $\gamma R = \beta$, then

$$a(\gamma) = a\left(\frac{\beta}{R}\right) = a_1(\beta); \qquad c(\gamma) = c\left(\frac{\beta}{R}\right) = c_1(\beta); \qquad w_0(R) = w_0^*(\rho);$$

$$\text{for } \rho < 1 \qquad \int_0^\infty c_1(\beta)a_1(\beta)\beta J_0(\beta\rho)\,d\beta = \frac{2R}{\pi} w_0^*(\rho), \tag{11}$$

$$\text{for } \rho > 1 \qquad \int_0^\infty a_1(\beta)\beta J_0(\beta\rho)\,d\beta = 0. \tag{12}$$

The solution of this system of equations for the case $c_1(\beta) = \beta^\alpha$ is known; it is of interest to obtain a solution of this system also for other functions $c(\gamma)$, which are Hankel transforms of other kernels K.

2. Let us consider the problem for a foundation whose modulus of elasticity varies with depth according to a power law. A foundation of this type was considered by G. K. Klein. He showed, among other things, that the stress distribution in this half-space, loaded by a concentrated force, coincides with the distribution proposed in the work of N. N. Ivanov, and after him in the work of O. Fröhlich.

In this case the kernel is

$$K(r) = \frac{1}{\pi E_n r^{n+1}},$$

and for $n > 0$ the modulus of elasticity increases with depth.

Let us examine this case:

$$c(\gamma) = 2\pi \int_0^\infty rK(r)J_0(\gamma r)\,dr = \frac{2^{1-n}}{E_n} \frac{\Gamma\left(\dfrac{1-n}{2}\right)}{\Gamma\left(\dfrac{1+n}{2}\right)} \gamma^{n-1}. \tag{13}$$

Let us introduce, for brevity of notation,

$$\frac{2^{1-n}}{E_n}\frac{\Gamma\left(\dfrac{1-n}{2}\right)}{\Gamma\left(\dfrac{1+n}{2}\right)}=A_n; \qquad n-1=\alpha;$$

$$\frac{2}{\pi}w_0^*\frac{R^n}{A_n}=g(\rho); \qquad \beta a_1(\beta)=f(\beta).$$

Then the system (11), (12) takes the form:

$$\int_0^\infty \beta^\alpha f(\beta)J_0(\beta\rho)\,d\beta=g(\rho) \qquad (0<\rho<1), \tag{14}$$

$$\int_0^\infty f(\beta)J_0(\beta\rho)\,d\beta=0 \qquad (1<\rho<\infty). \tag{15}$$

Put $1>n>0$; then $\alpha<0$, and the solution of the problem can be obtained by using the result of Busbridge (5), which for zero index of the Bessel function reduces to the formula:

$$\begin{aligned} f(\beta)= & \frac{2^{-\frac{1}{2}\alpha}}{\Gamma\left(1+\frac{1}{2}\alpha\right)}\frac{\beta^{-\alpha}}{}\Bigg[\beta^{(1+\frac{1}{2}\alpha)}J_{\frac{1}{2}\alpha}(\beta)\int_0^1 y(1-y^2)^{\frac{1}{2}\alpha}g(y)\,dy \\ & +\int_0^1 u(1-u^2)^{\frac{1}{2}\alpha}\,du\int_0^1 g(yu)(\beta y)^{(2+\frac{1}{2}\alpha)}J_{(1+\frac{1}{2}\alpha)}(\beta y)\,dy\Bigg]. \end{aligned} \tag{16}$$

If we put $g=g_0=\text{const}$, i.e. assume that the stamp has a plane lower boundary, then as a result of the calculations we obtain:

$$f(\beta)=g_0\frac{2^{\frac{1-n}{2}}\beta^{\frac{1-n}{2}}}{\Gamma\left(\dfrac{n+1}{2}\right)}J_{\frac{n+1}{2}}(\beta). \tag{17}$$

The reactive pressure is equal to

$$p(\rho)=-\frac{\pi}{2}g_0\frac{2^{\frac{n+1}{2}}}{\Gamma\left(\dfrac{n+1}{2}\right)}\frac{1}{R^2}\int_0^\infty \beta^{-\frac{n+1}{2}}J_{\frac{n+1}{2}}(\beta)J_0(\beta\rho)\,d\beta. \tag{18}$$

Now the problem reduces to evaluating an improper Weber–Schafheitlin integral, which in our case gives:

$$p(\rho) = -\frac{\pi}{n+1}\frac{g_0}{\rho}\frac{1}{R^2}\frac{{}_2F_1\left(\frac{1}{2};\frac{1}{2};\frac{3+n}{2};\frac{1}{\rho^2}\right)}{\Gamma^2\left(\frac{n+1}{2}\right)}. \tag{19}$$

Of greatest interest is the determination of the pressures at the center and at the edge of the stamp. The first of these reduces to the evaluation of an integral of Weber type, and the second—to an integral of Struve type. After calculation we obtain:

$$p(0) = -\frac{\pi\sqrt{\pi}}{2} g_0 \frac{1}{R^2}\frac{1}{\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(1+\frac{n}{2}\right)}, \tag{20}$$

$$p(1) = -\frac{\pi}{2} g_0 \frac{1}{R^2}\frac{2}{\Gamma^2\left(1+\frac{n}{2}\right)}. \tag{21}$$

3. It is not difficult to apply the same scheme to the solution of the problem of a plane punch. In this case, as follows from the formulas given at the beginning of this article, the problem reduces to solving the system of equations

$$\int_0^\infty y^\alpha f_1(y)\cos xy\,dy = g(x) \qquad (0 < x < 1), \tag{22}$$

$$\int_0^\infty f_1(y)\cos xy\,dy = 0 \qquad (1 < x < \infty), \tag{23}$$

which can also be regarded as dual equations for the case of a Bessel function of half-integer index. In this case, putting, for example,

$$\cos xy = J_{\frac{1}{2}}(xy)e^{xy}$$

and denoting $e^{xy} f_1(y) = f(y)$, we obtain the system:

$$\int_0^\infty y^\alpha f(y) J_{\frac{1}{2}}(xy)\,dy = g(x) \qquad (0 < x < 1), \tag{24}$$

$$\int_0^\infty f(y) J_{\frac{1}{2}}(xy)\,dy = 0 \qquad (1 < x < \infty). \tag{25}$$

It is comparatively easy to generalize the result to the case of a punch with a non-flat boundary:

$$W(r) = \sum_{m=1}^{k} \varphi_m(r) \cos m\varphi. \tag{26}$$

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CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.