TWO EXPANSION THEOREMS CONNECTED WITH BOUNDARY-VALUE PROBLEMS FOR THE EQUATION $\psi_{\sigma\sigma}-K(\sigma)\psi_{\theta\theta}=0$

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Abstract

Full Text

MATHEMATICS

R. G. BARANTSEV

TWO EXPANSION THEOREMS CONNECTED WITH BOUNDARY-VALUE PROBLEMS FOR THE EQUATION $\psi_{\sigma\sigma} - K(\sigma)\psi_{\theta\theta} = 0$

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1. Let, in the strip $S\{\sigma_0 \leq \sigma \leq \sigma_1,\ -\infty < \theta < +\infty\}$, $K(\sigma) \geq \varepsilon > 0$, and let $\theta = s(\sigma)$ be a certain curve s with endpoints at the points (σ_0, θ_0), (σ_1, θ_1). Consider the problem

$$\left.\begin{array}{c} \psi_{\sigma\sigma} - K(\sigma)\psi_{\theta\theta} = 0; \qquad \psi|_s = \bar{\psi}(\sigma); \\ \psi_\theta|_s \begin{cases} = \bar{\psi}_1(\sigma), & \text{if } s \text{ is oriented along the } \sigma\text{-axis}, \\ \text{is not prescribed}, & \text{if } s \text{ is a characteristic}; \end{cases} \\ \psi(\sigma_0, \theta)a_0 + \psi_\sigma(\sigma_0, \theta)b_0 = 0, \qquad \psi(\sigma_1, \theta)a_1 + \psi_\sigma(\sigma_1, \theta)b_1 = 0, \end{array}\right\} \qquad (C_0)$$

where $\bar{\psi}(\sigma)$, $\bar{\psi}_1(\sigma)$ are given functions; a_0, b_0, a_1, b_1 are constants such that $a_0^2 + b_0^2 \neq 0$, $a_1^2 + b_1^2 \neq 0$.

By replacing (σ, θ) by (x, t) and ψ by v according to the formulas

$$cx = \int_{\sigma_0}^{\sigma} \sqrt{K}\, d\sigma, \qquad ct = \theta - \theta_0, \qquad c = \int_{\sigma_0}^{\sigma_1} \sqrt{K}\, d\sigma, \qquad v(x,t) = \psi(\sigma, \theta)K^{1/4}(\sigma),$$

problem (C_0) is reduced to the form ([1])

$$\left.\begin{array}{c} v_{xx} - v_{tt} + N(x)v = 0; \qquad v|_{t=l(x)} = p(x); \\ v_t|_{t=l(x)} \begin{cases} = q(x), & \text{if } |l'(x)| < 1, \\ \text{is not prescribed}, & \text{if } l(x) \equiv \pm x; \end{cases} \\ v(0,t)\cos\alpha + v_x(0,t)\sin\alpha = 0, \qquad 0 < \alpha \leq \pi; \\ v(1,t)\cos\beta + v_x(1,t)\sin\beta = 0, \qquad 0 < \beta \leq \pi, \end{array}\right\} \qquad \text{(C)}$$

where

$$N(x) = -K^{-1/4}(\sigma)\frac{d^2K^{1/4}(\sigma)}{dx^2}.$$

2. Let s_n and $B_n(x)$ be the eigenvalues and normalized eigenfunctions of the following Sturm-Liouville problem:

$$B_n'' + [s_n + N(x)]B_n = 0, \tag{1}$$

$$B_n(0)\cos\alpha + B_n'(0)\sin\alpha = 0, \qquad B_n(1)\cos\beta + B_n'(1)\sin\beta = 0. \tag{2}$$

Put $\lambda_{\pm n} = \pm\sqrt{s_n}$. We assume that zero is not an eigenvalue.

If the solution of problem (C) is sought in the form

$$v = \sum_{n=-\infty}^{\infty}{}' c_n B_n(x)\exp(-i\lambda_n t), \tag{3}$$

then the satisfaction of the initial conditions on l leads to the expansion of $p(x)$ and $q(x)$ in the series

$$p(x) \approx \sum_{n=-\infty}^{\infty}{}' c_n \bar{z}_n(x), \tag{4}$$

$$q(x) \approx \sum_{n=-\infty}^{\infty}{}' c_n(-i\lambda_n)\bar{z}_n(x), \tag{5}$$

where

$$\bar{z}_n(x) = B_n(x)\exp[-i\lambda_n l(x)]. \tag{6}$$

Substituting $B_n(x)$ from (6) into (1) and (2), we obtain for $\bar{z}_n(x)$ the equation

$$\bar{z}_n'' + 2i\lambda_n l'\bar{z}_n' + \bar{z}_n\{N + i\lambda_n l'' + \lambda_n^2(1 - l'^2)\} = 0 \tag{7}$$

and the boundary conditions

$$\begin{aligned} &\bar{z}_n(0)[\cos\alpha + i\lambda_n l'(0)\sin\alpha] + \bar{z}_n'(0)\sin\alpha = 0, \\ &\bar{z}_n(1)[\cos\beta + i\lambda_n l'(1)\sin\beta] + \bar{z}_n'(1)\sin\beta = 0. \end{aligned} \tag{8}$$

Thus, in order to determine the coefficients c_n of the series (3), it is necessary to expand $p(x)$ and $q(x)$ in the form (4), (5) in terms of the eigenfunctions of the non-self-adjoint system (7), (8).

3. Writing the equation and boundary conditions for the function $z_n(x) = B_n(x)\exp[i\lambda_n l(x)]$, analogous to (7), (8), it is easy to obtain from (3) the following orthogonality relation:

$$I_{m,n} \equiv \int_0^1 \{l'(z_n'\bar z_m - \bar z_n z_m') + i(1-l'^2)(\lambda_n + \bar\lambda_m) z_n \bar z_m\}\,dx = 0, \qquad m \neq n. \quad (9)$$

With the help of (9), also using the fact that $I_{n,n} = 2i\lambda_n$, the coefficients c_n of the series (3) can formally be determined in the form

$$c_n = \frac{i}{2\lambda_n}\int_0^1 \{-p\,l'' z_n + z_n[p'l' + (1-l'^2)(q - i\lambda_n p)]\}\,dx, \qquad |l'(x)| < 1; \quad (10)$$

$$c_n = \frac{i}{2\lambda_n}[pz_n']_0^1 - \frac{i}{\lambda_n}\int_0^1 pz_n'\,dx, \qquad l(x) \equiv x. \quad (11)$$

4. Let us denote the partial sums of the series (4), (5) with coefficients (10), respectively, by $S_n^{(p)}(x)$ and $S_n^{(q)}(x)$, and the partial sum of the series (4) with coefficients (11) by $\Sigma_n^{(p)}(x)$.

Theorem 1. If on $[0,1]$ $|l'(x)| < 1$ and the functions $N(x)$, $l''(x)$, $p'(x)$, $q(x)$ have bounded variation, then as $n \to \infty$:

in case A $(0 < \alpha < \pi,\ 0 < \beta < \pi)$

$$S_n^{(p)}(x) \to p(x), \qquad 0 \le x \le 1;$$

$$S_n^{(q)}(x) \to \frac{q(x-0) + q(x+0)}{2}, \qquad 0 < x < 1;$$

$$S_n^{(q)}(0) \to q(0+), \qquad S_n^{(q)}(1) \to q(1-);$$

in case B $(\alpha = \beta = \pi)$

$$S_n^{(p)}(x) \to p(x), \qquad 0 < x < 1;$$

$$S_n^{(q)}(x) \to \frac{q(x-0) + q(x+0)}{2}, \qquad \text{if } p(0)l(x) = p(1)[l(1) - l(x)] = 0,$$

$$0 < x < 1;$$

$$S_n^{(p)}(0) = S_n^{(p)}(1) = S_n^{(q)}(0) = S_n^{(q)}(1) = 0;$$

in case C $(\alpha = \pi,\ 0 < \beta < \pi)$

$$S_n^{(p)}(x) \to p(x), \qquad 0 < x \leqslant 1;$$

$$S_n^{(q)}(x) \to \frac{q(x-0)+q(x+0)}{2}, \qquad \text{if } p(0)l(x) = 0, \quad 0 < x < 1;$$

$$S_n^{(q)}(1) \to q(1-), \qquad \text{if } p(0)l(1) = 0;$$

$$S_n^{(p)}(0) = S_n^{(q)}(0) = 0;$$

in case D $(0 < \alpha < \pi,\ \beta = \pi)$

$$S_n^{(p)}(x) \to p(x), \qquad 0 \leqslant x < 1;$$

$$S_n^{(q)}(x) \to \frac{q(x-0)+q(x+0)}{2}, \qquad \text{if } p(1)[l(1)-l(x)] = 0, \quad 0 < x < 1;$$

$$S_n^{(q)}(0) \to q(0+), \qquad \text{if } p(1)l(1) = 0;$$

$$S_n^{(p)}(1) = S_n^{(q)}(1) = 0.$$

This theorem generalizes Langer' s results ([2]). In case B, for $p(0) = p(1) = 0$, it was proved in ([3]).

Theorem 2. If $l(x) \equiv x$ and the functions $N(x)$, $p(x)$ have bounded variation on $[0,1]$, then as $n \to \infty$:

in case A $(0 < \alpha < \pi,\ 0 < \beta < \pi)$

$$\Sigma_n^{(p)}(x) \to \frac{p(x-0)+p(x+0)}{2}, \qquad 0 < x < 1;$$

$$\Sigma_n^{(p)}(0) \to p(0+), \qquad \Sigma_n^{(p)}(1) \to p(1-);$$

in case B $(\alpha = \beta = \pi)$

$$\Sigma_n^{(p)}(x) \to \frac{p(x-0)+p(x+0)-p(0+)-p(1-)}{2}, \qquad 0 < x < 1;$$

$$\Sigma_n^{(p)}(0) = \Sigma_n^{(p)}(1) = 0;$$

in case C $(\alpha = \pi,\ 0 < \beta < \pi)$

$$\Sigma_n^{(p)}(x) \to \frac{p(x-0) + p(x+0) - p(0+)}{2}, \qquad 0 < x < 1;$$

$$\Sigma_n^{(p)}(0) = 0, \qquad \Sigma_n^{(p)}(1) \to p(1-) - p(0+);$$

in case D $(0 < \alpha < \pi,\ \beta = \pi)$

$$\Sigma_n^{(p)}(x) \to \frac{p(x-0) + p(x+0) - p(1-)}{2}, \qquad 0 < x < 1;$$

$$\Sigma_n^{(p)}(0) \to p(0+) - p(1-), \qquad \Sigma_n^{(p)}(1) = 0.$$

For $\alpha = \beta = \pi$ (case B) we have an intersection with the results of Mishoe ([4]). The unusual fact noted in ([4]), that the sum of the expansion of $p(x)$ at any point $x \in [0,1]$ depends, generally speaking, on $p(0+)$ and $p(1-)$, occurs, as is seen from Theorem 2, only in those cases when α or β is equal to π.

5. The proof of the theorems is carried out, as in ([3]), by the method of contour integration in the complex λ-plane over an infinitely large circle. With the aid of a modification of the generalized Fourier transform ([3]), one can show that

$$S_n^{(p)}(x) = \frac{1}{2\pi} \oint_{C_n} \Phi(x,\lambda)\, d\lambda, \qquad S_n^{(q)}(x) = -\frac{i}{2\pi} \oint_{C_n} \lambda \Phi(x,\lambda)\, d\lambda,$$

where

$$\Phi(x,\lambda) =$$

$$= e^{-i\lambda l(x)} \left\{ \frac{\chi(x,\lambda)}{\omega(\lambda)} \int_0^x e^{i\lambda l(y)} \left[\varphi(y,\lambda)(p'l' + q(1-l'^2) - i\lambda p) - \varphi'(y,\lambda) p l'\right] dy + \right.$$

$$\left. + \frac{\varphi(x,\lambda)}{\omega(\lambda)} \int_x^1 e^{i\lambda l(y)} \left[\chi(y,\lambda)(p'l' + q(1-l'^2) - i\lambda p) - \chi'(y,\lambda) p l'\right] dy \right\};$$

$$\Sigma_n^{(p)}(x) = -\frac{1}{\pi}\oint_{C_n} \Psi(x,\lambda)\, d\lambda;$$

$$\Psi(x,\lambda) = e^{-i\lambda x}\left\{\frac{p(0+)}{2}\frac{\chi(x,\lambda)}{\omega(\lambda)}\sin\alpha - \frac{p(1-)}{2}\frac{\varphi(x,\lambda)}{\omega(\lambda)}e^{i\lambda}\sin\beta + \right.$$

$$\left. + \frac{\chi(x,\lambda)}{\omega(\lambda)}\int_0^x e^{i\lambda y} p(\varphi' + i\lambda\varphi)\, dy + \frac{\varphi(x,\lambda)}{\omega(\lambda)}\int_x^1 e^{i\lambda y} p(\chi' + i\lambda\chi)\, dy\right\}.$$

Here $\varphi(x,\lambda)$, $\chi(x,\lambda)$ are solutions of equation (1), where $s = \lambda^2$, such that

$$\varphi(0,\lambda) = \sin\alpha, \qquad \varphi'(0,\lambda) = -\cos\alpha;$$

$$\chi(1,\lambda) = \sin\beta, \qquad \chi'(1,\lambda) = -\cos\beta;$$

$$\omega(\lambda) = \varphi\chi'_x - \chi\varphi'_x.$$

C_n is a circle with center at the origin and radius R_n, satisfying the inequality $\lambda_n + \varepsilon \le R_n \le \lambda_{n+1} - \varepsilon$, $\varepsilon > 0$.

In passing to the limit ($n \to \infty$), essential use is made of the asymptotic formulas for the functions $\varphi, \chi, \varphi', \chi', \omega$, obtained by the method indicated in (5) (§1.7).

Leningrad State University
named after A. A. Zhdanov

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