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Abstract

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ON THE FUNCTION OF THE ANGULAR AND SPATIAL DISTRIBUTION OF PARTICLES AT THE MAXIMUM OF A CASCADE SHOWER

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The problem of the one-dimensional development of an electron-photon cascade shower may be regarded as completely solved. The method of functional transformations ([1]) and the method of moments ([2]) make it possible to describe with sufficient accuracy the longitudinal development of the mean shower in light and heavy substances.

The situation is quite different with the solution of the problem of the three-dimensional development of a cascade shower. An analytical expression for the function of the spatial distribution* has not yet been found, owing to the extraordinary complexity of the initial equations; for the function of the angular distribution, an analytical expression had been found only without allowance for ionization losses ([1]). A number of authors have carried out numerical calculations of the angular- and spatial-distribution functions for various particular cases (see, for example, ([3])), but these calculations are so complicated that even an approximate scheme of the computations has not yet been published.

It seems to us that a promising method for solving the problem of the angular and spatial distribution of particles in a shower is the method of moments. In ([4,5]), the angular- and spatial-distribution function of particles was obtained for an arbitrary depth by the method of moments. In these works the distribution functions were constructed from the moments by graphically selecting an arbitrary function whose first few moments coincided with the exact ones. In addition, the authors of ([4,5]) did not take into account the pole of the spatial-distribution function, and therefore they did not succeed in constructing this function in the neighborhood of zero.

One may propose a simple method for calculating the angular- and spatial-distribution function at the maximum of a cascade shower from the known moments.

From an analysis of the equations for the angular- and spatial-distribution functions it is easy to obtain that these functions depend on $\bar{r}$ and $\bar{\theta}$ only in the combination $x_\theta = \dfrac{E\theta}{E_s}$ $\left(x_r = \dfrac{Er}{E_s}\right)$, where $E_s = 21$ MeV. It has been shown that the differential angular-distribution function

$$P(E_0, E, \theta, t) = P_{\text{long}}(E_0, E, t)\, P(s, x_\theta) \tag{1}$$

remains finite at $\theta = 0$ ([1]). We approximate the function $P(1, x_\theta)$ by means of a sum of polynomials $V_n(\alpha x)$:

$$P(1, x_\theta) = e^{-\alpha x_\theta} \sum_{n=0}^{k/2-1} a_n V_n(\alpha x_\theta). \tag{2}$$

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* Throughout this paper, by the angular-distribution function we mean the function integrated over the plane perpendicular to the shower axis; by the spatial-distribution function, the function integrated over angles.

The polynomials $V_n(\alpha x)$ are determined, up to a normalization constant, from the condition

$$\int_0^\infty x^{2n'+1} e^{-\alpha x} V_n(\alpha x)\, dx = 0 \quad \text{for } n' < n. \tag{3}$$

Their explicit expressions are

$$V_0(y) = \alpha; \qquad V_1(y) = \frac{\alpha}{2}(-2 + y); \qquad V_2(y) = \frac{\alpha}{8}(8 - 7y + y^2);$$

$$V_3(y) = \frac{\alpha}{48}(-48 + 57y - 15y^2 + y^3).$$

The polynomials orthogonal to $V_n(y)$ are determined from the equality

$$\int_0^\infty e^{-\alpha x} V_n(\alpha x) V_{n'}^+(\alpha x)\, dx = \delta_{nn'}. \tag{4}$$

Their explicit expressions are:

$$V_0^+(y) = y; \qquad V_1^+(y) = -y + \frac{y^3}{3!}; \qquad V_2^+(y) = y - 2\frac{y^3}{3!} + \frac{y^5}{5!};$$

$$V_3^+(y) = -y + 3\frac{y^3}{3!} - 3\frac{y^5}{5!} + \frac{y^7}{7!}.$$

Using the polynomials $V_n^+(y)$, it is easy to find the following expressions for the coefficients a_n:

$$a_n = \int_0^\infty V_n^+(\alpha x) P(1, x_\theta)\, dx; \qquad (5)$$

$$a_0 = 1; \qquad a_1 = -\frac{3!}{\alpha^2}\left(1 - \alpha^2 \frac{\overline{\theta^2}}{3!}\right), \qquad a_2 = \frac{5!}{\alpha^4}\left(1 - 3\alpha^2\frac{\overline{\theta^2}}{3!} + \alpha^4 \frac{\overline{\theta^4}}{5!}\right);$$

$$a_3 = -\frac{7!}{\alpha^6}\left(1 - 3\alpha^2\frac{\overline{\theta^2}}{3!} + 3\alpha^4\frac{\overline{\theta^4}}{5!} - \alpha^6\frac{\overline{\theta^6}}{7!}\right).$$

Thus, the coefficients a_n are simply expressed in terms of the corresponding moments of the angular-distribution function. We find the coefficient α from the following condition: let us require that the k-th moment of the weight function $e^{-\alpha x}$ be equal to the k-th moment of the desired distribution function, i.e. we determine α from the condition

$$\int_0^\infty e^{-\alpha x} x^k\, dx \Big/ \int_0^\infty e^{-\alpha x}\, dx = \overline{\theta^k}. \qquad (6)$$

Table 1*

x_θ	$P(1, x_\theta)$, $k = 6$	$P(1, x_\theta)$, $k = 4$	$P(1, x_\theta)$, from (4)	x_θ	$P(1, x_\theta)$, $k = 6$	$P(1, x_\theta)$, $k = 4$	$P(1, x_\theta)$, from (4)
0	9.84	10.42	9.3	1.6	$6.06 \cdot 10^{-2}$	$5.75 \cdot 10^{-2}$	$6.00 \cdot 10^{-2}$
0.2	5.30	5.49	5.3	1.8	3.10	3.03	2.90
0.4	2.85	2.89	2.78	2.0	1.56	1.56	1.46
0.6	1.51	1.51	1.52	2.2	$7.76 \cdot 10^{-3}$	$8.31 \cdot 10^{-3}$	$7.13 \cdot 10^{-3}$
0.8	$8.6 \cdot 10^{-1}$	$7.93 \cdot 10^{-1}$	$8.20 \cdot 10^{-1}$	2.4	3.76	4.32	3.56
1.0	4.28	4.14	4.46	2.6	1.76	2.24	1.74
1.2	2.25	2.16	2.32	2.8	$7.92 \cdot 10^{-4}$	1.15	$8.29 \cdot 10^{-4}$
1.4	1.17	1.12	1.18	3.0	3.28	$6.02 \cdot 10^{-4}$	3.92

* In calculating the functions $P(1, x_\theta)$ and $P(1, x_r)$, the numerical values of the moments given in papers ([4,5]) were used.

The results of calculations of the function $P(1, x_\theta)$ are given in Table 1. The first column gives the values of $P(1, x_\theta)$ calculated by formula (2) for $k = 6$, i.e., using the first three even moments; the second column gives the values calculated for $k = 4$; the third column gives those from the data of ([4]). From an analysis of Table 1 it is seen that the proposed form of approximation of the function $P(1, x_\theta)$, even for $k = 4$, i.e., using the first two even moments, makes it possible to obtain, with good accuracy, the values of $P(1, x_\theta)$ over a wide range of values of x_θ. Let us note that comparison of the values of $P(1, x_\theta)$ for $k = 6$ and $k = 4$ makes it possible to determine the limits of applicability of approximation (2) for a given value of k.

To obtain an approximation formula for the integral function of the spatial distribution

$$N(E_0, E, r, t) = N_{\text{prod}}(E_0, E, t)\, N(s, x_r)$$

at the shower maximum, we proceed as follows. In ([3], [6], [7]) it is shown that as $r \to 0$ the function $N(s, x_r)$ is proportional to r^{s-2}. We represent the function $N(s, x_r)$ at the shower maximum in the form

$$N(1, x_r) = \frac{1}{x_r}\, P(1, x_r). \tag{7}$$

Let us note that the moments of the functions $N(E_0, E, r, t)$ and $P(1, x_r)$ are related by the equality $\overline{r^n} = \left(\frac{E_s}{E}\right)^n \overline{x_p^n}$.

We approximate the function $P(1, x_r)$ by means of a sum of polynomials $R_n(\alpha x)$:

$$P(1, x_r) = e^{-\alpha x_r^{1/2}} \sum_{n=0}^{k/2-1} a_n R_n(\alpha x_r). \tag{8}$$

The polynomials $R_n(\alpha x)$ are determined from the condition

$$\int_0^\infty x^{2n'} e^{-\alpha x^{1/2}} R_n(\alpha x)\, dx = 0 \quad \text{for } n' < n.$$

Their explicit expressions are:

$$R_0(y) = \frac{\alpha^2}{2}; \qquad R_1 = \frac{\alpha^2}{8640}\left(-6 + \alpha^2 x\right);$$

$$R_2(y) = \frac{\alpha^2}{1.8835 \cdot 10^9}\left(1092 - 242\,\alpha^2 x + 3\alpha^4 x^2\right).$$

The polynomials, orthogonal $R_n(y)$, are determined from condition (6) with weight function $e^{-\alpha x^{1/2}}$ instead of $e^{-\alpha x}$. Their explicit expressions are:

$$R_0^+(y) = 1; \qquad R_1^+(y) = -120+\alpha^4 x^2; \qquad R_2^+(y) = 6.854\cdot 10^5-8.736\cdot 10^3\alpha^4 x^2+\alpha^8 x^4.$$

Using formula (5), one can find the following expressions for the coefficients a_n:

$$a_0 = 1; \qquad a_1 = -120 + \alpha^4 \overline{x_p^2}; \qquad a_2 = 6.854 \cdot 10^5 - 8.736 \cdot 10^3 \alpha^4 \overline{x_p^2} + \alpha^8 \overline{x_p^4}.$$

The coefficient α in (8) is determined from the condition

$$\int_0^\infty e^{-\alpha x^{1/2}} x^k \, dx \Big/ \int_0^\infty e^{-\alpha x^{1/2}} \, dx = \overline{x_p^k}.$$

The results of calculations of the function $P(1, x_r)$ are given in Table 2. The first column gives the values of $P(1, x_r)$ calculated by formula (8) for $k = 6$; the second gives those calculated for $k = 4$; the third gives the data of paper ([5]). From an analysis of Table 2 it is seen that the proposed form of the approximation, even for $k = 4$, i.e., using two even moments,

Table 2*

x_r	$\frac{x_r}{E}P(1,x_r)$, $k=6$	$\frac{x_r}{E}P(1,x_r)$, $k=4$	$\frac{x_r}{E}P(1,x_r)$, from ([5])	x_r	$\frac{x_r}{E}P(1,x_r)$, $k=6$	$\frac{x_r}{E}P(1,x_r)$, $k=4$	$\frac{x_r}{E}P(1,x_r)$, from ([5])
0	10,77	11,18	–	0,15	1,80	–	–
0,0001	10,28	–	–	0,2	1,36	1,35	1,62
0,0002	10,09	–	–	0,6	$2,93 \cdot 10^{-1}$	–	$3,16 \cdot 10^{-1}$
0,0005	9,72	–	–	1,0	1,01	$9,91 \cdot 10^{-2}$	1,03
0,001	9,31	–	–	1,4	$4,24 \cdot 10^{-2}$	–	$4,11 \cdot 10^{-2}$
0,002	8,77	–	–	1,8	1,99	–	1,86
0,005	7,78	–	–	2,2	1,01	–	$9,34 \cdot 10^{-3}$
0,010	6,80	–	–	2,6	$5,46 \cdot 10^{-3}$	–	5,12
0,015	6,13	–	–	3,0	3,06	$3,13 \cdot 10^{-3}$	3,02

x_r	$\frac{x_r}{E}P(1,x_r)$, $k=6$	$\frac{x_r}{E}P(1,x_r)$, $k=4$	$\frac{x_r}{E}P(1,x_r)$, from ([5])	x_r	$\frac{x_r}{E}P(1,x_r)$, $k=6$	$\frac{x_r}{E}P(1,x_r)$, $k=4$	$\frac{x_r}{E}P(1,x_r)$, from ([5])
0,02	5,62	–	–	4,0	$8,45\cdot10^{-4}$	$8,88\cdot10^{-4}$	1,09
0,04	4,47	–	–	5,0	2,72	2,93	–
0,06	3,48	–	–	6,0	1,01	1,07	–
0,1	2,50	–	–	7,0	$3,89\cdot10^{-5}$	$4,25\cdot10^{-5}$	–

* In paper ([5]) the differential function $P(1,x_r)$ was calculated in the interval $0.2 < x_r < 6$. Therefore, in order to obtain the integral function, we assumed that the function $P(1,x_r)$ in ([5]) over the interval $(6,\infty)$ is proportional to e^{-x_r}/x_r. We note that the addition obtained in this way changes the last digit of the corresponding column by 40%, and its effect on the remaining digits is still smaller.

makes it possible to obtain $P(1,x_r)$ with good accuracy over a wide range of values of x_r. As in the case of the angular distribution, comparison of the data for $k=6$ and $k=4$ makes it possible to determine the limits of applicability of approximation (8) for a given value of k.

We hope that by the proposed method one can obtain the functions of the angular and spatial distribution with allowance for ionization losses. The corresponding calculations are being carried out.

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Note: Figure translations are in progress. See original paper for figures.