ON THE APPROXIMATION OF ABSTRACT CONTINUOUS FUNCTIONS BY UNBOUNDED OPERATOR-FUNCTIONS

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1957

SovietRxiv

Abstract

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MATHEMATICS

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(Presented by Academician N. N. Bogolyubov, 26 IV 1957)

1. Let a system of linear differential equations be given

$$\frac{dx_i}{dt} + \sum_{k=1}^{n} p_{ik}(t)x_k = f_i(t) \qquad (0 \le t \le 2\pi) \quad (i = 1, 2, \dots, m) \tag{1}$$

and suppose that one seeks such a solution of this system which satisfies certain boundary conditions, for example $x_k(0) = x_k(2\pi)$ $(k = 1, \dots, n)$, and belongs to a prescribed class D of vector-functions $x = (x_1(t), x_2(t), \dots, x_n(t))$.

If in the class D there is no such (exact) solution, then one may pose the question of finding in it the best approximate solution of system (1), i.e. such a vector-function $x_0 = (x_{10}(t), x_{20}(t), \dots, x_{n0}(t))$ that

$$\max_i \left\| \frac{dx_{i0}}{dt} + \sum_{k=1}^{n} p_{ik}(t)x_{k0} - f_i(t) \right\| = \inf_{x \in D} \max_i \left\| \frac{dx_i}{dt} + \sum_{k=1}^{n} p_{ik}(t)x_k - f_i(t) \right\|.$$

For each $i = 1, 2, \dots, m$, the left-hand side of the i-th equation of system (1) may be regarded as the value of an operator A_i, acting from the Hilbert space H of vector-functions $x = (x_1(t), x_2(t), \dots, x_n(t))$ (where

$$x_k(t) \in L^2(0, 2\pi) \quad (k = 1, 2, \dots, n) \quad \text{and} \quad \|x\|_H = \left(\sum_{k=1}^{n} \|x_k\|_{L^2}^2 \right)^{1/2}$$

) into $L^2(0, 2\pi)$.

The same problem can be posed when one is dealing, in general, with a system of operator equations. More precisely, suppose that on some compact set Q there is considered an operator-function $A(q)$, which for each $q \in Q$ is a linear operator acting from a Hilbert space H_1 into a Hilbert space H_2, with a domain of definition D common to all $q \in Q$, and such that for each fixed $x \in H_1$ the function $A(q)x$, with values in H_2, is continuous on Q. Let $f(q)$ be a function continuous on Q with values in H_2. The problem consists in finding a vector $x_0 \in D$ such that the function $A(q)x_0$ deviates least on Q from the function $f(q)$, i.e. so that*

$$\max_{q \in D} \|A(q)x_0 - f(q)\|_2 = \inf_{x \in D} \max_{q \in Q} \|A(q)x - r(q)\|_2.$$

* We shall denote the norm in H_1 by the subscript 1, and in H_2 by the subscript 2; the zero in H_1 will be denoted by θ_1, and in H_2 by θ_2.

Let R be the subspace of those vectors $x \in D$ for which $A(q)x = \theta_2$ for all $q \in Q$, and let S be the orthogonal complement of R in H_1 (for the operator-functions considered below, the linear manifold R will turn out to be a subspace); then each vector $x \in D$ is represented in the form $x = x_R + x_S$ ($x_S \in D \cap S$), and

$$\inf_{x \in D} \max_{q \in Q} \|A(q)x - f(q)\|_2 = \inf_{x_S \in D \cap S} \max_{q \in Q} \|A(q)x_S - f(q)\|_2.$$

Therefore in what follows, without stipulating this, we shall assume that $R = \theta_1$ and $S = H_1$.

In (1) an analogous problem was considered only for the case when the operator-function $A(q)$, for each $q \in Q$, was a linear bounded operator acting from the Hilbert space H into the same space. In the present paper we consider linear unbounded (closed) operators; we prove that property a), given in (1), is not only a sufficient but also a necessary condition for the existence of a least-deviating function both in the case of bounded and in the case of closed operators; we consider questions of uniqueness of the function of least deviation and extend some of the results obtained to Banach spaces.

2. Theorem 1. *Let an operator-function $A(q)$ be given on the compact set Q, possessing the following two properties: 1) for each $q \in Q$, $A(q)$ is a closed linear operator acting from the Hilbert space H_1 into the Hilbert space H_2, with a domain of definition $D_{A(q)} = D$ dense in H_1 and common to all $q \in Q$; 2) for each fixed $x \in D$, $A(q)x$ is a continuous function on Q with values in H_2.*

Then, in order that for every function $f(q)$, continuous on Q and with values in H_2, there exist a vector $x_0 \in D$ such that the function $A(q)x_0$ deviates least on Q from the function $f(q)$:

$$\inf_{x \in D} \max_{q \in Q} \|A(q)x - f(q)\|_2 = \max_{q \in Q} \|A(q)x_0 - f(q)\|_2,$$

it is necessary and sufficient that $A(q)$ possess the following property:

$$\max_{q \in Q} \|A(q)x\|_2 \geq m\|x\|_1 \quad \text{for all } x \in D, \tag{a}$$

where $m > 0$.

The **necessity** of property (a) is proved as follows. The functions $f(q)$, continuous on Q and with values in H_2, form a Banach space C with norm $\|f\|_C = \max_{q\in Q}\|f(q)\|_2$, and, obviously, in order that for every function $f(q)$ there exist a function $A(q)x_0$ deviating least from it, it is necessary that the linear manifold of functions $A(q)x$ $(x \in D)$ be closed in C, i.e. form a subspace. Further, the operator T, acting from H_1 into C according to the rule $Tx = A(q)x$ $(x \in D)$, is closed, since the operators $A(q)$ are closed for each $q \in D$; moreover, the operator T has an inverse T^{-1}, since, by assumption, $R = \theta_1$. But this inverse operator T^{-1} is bounded, being closed and defined on a subspace (see, for example, (^{2}), p. 47), so that

$$\|Tx\|_C = \max_{q\in Q}\|A(q)x\|_2 \geq \|x\|_1 \quad (x \in D),$$

where $m > 0$.

The **sufficiency** of property (a) is proved with more difficulty. We only note that it also essentially uses the closedness of the operator $A(q)$ and one theorem of S. Mazur (see (^{3}), p. 207), on which the proof of Theorem 1 in (^{1}) was also based.

3. Examples of the realization of the conditions of the preceding theorem, besides the operator-functions I–IV of work (^{1}), which are bounded operators for every $q \in Q$, may also be furnished by the following operator-functions:

V.

$$A(q)x = \sum_{i=1}^{N} \xi_i f_i(q),$$

where $f_1(q), f_2(q), \dots, f_N(q)$ are functions continuous on Q with values in the Hilbert space H, and $x = (\xi_1, \xi_2, \dots, \xi_N)$ is a point of a real or complex Euclidean space R_N; the question is that of approximation of abstract functions $f(q)$ by polynomials

$$\sum_{i=1}^{N} \xi_i f_i(q),$$

considered in (^{4}). Here $H_1 = R_N$, $H_2 = H$, and the operators $A(q)$ are bounded.

In particular, if $f_1(q), f_2(q), \dots, f_N(q)$ are real or complex continuous functions on Q, then $H = R_1$, and the question is that of the classical problem of Chebyshev approximation of a numerical function $f(q)$, continuous on Q, by a polynomial composed of numerical functions $f_1(q), f_2(q), \dots, f_N(q)$ continuous on Q.

VI. $A(q) = P_1 + \lambda(q)E$, where $P_1 = id/dt$ is the differentiation operator on $[0, 2\pi]$ with boundary conditions $x(0) = x(2\pi)$, and $\lambda(q)$ is a function continuous on Q, and at least for one $q_0 \in Q$ we have $\lambda(q_0) \neq 0, \pm 1, \pm 2, \ldots$. Here $H_1 = H_2 = L^2(0, 2\pi)$, and the operators $A(q)$ are unbounded.

VII. For every $q \in Q$, $A(q)$ is the Sturm–Liouville operator

$$A(q)x = \frac{d}{dt}\left[P_1(q,t)\frac{dx}{dt}\right] + p_2(q,t)x(t) \qquad (0 \le t \le 2\pi),$$

depending continuously on q, with boundary conditions $x(0) = x(2\pi) = 0$, and for some $q_0 \in Q$ zero is a regular point of the spectrum of the operator $A(q_0)$. Here $H_1 = H_2 = L^2(0, 2\pi)$, and the operators $A(q)$ are also unbounded.

In general, obviously, property (a) is possessed by every operator-function $A(q)$ satisfying conditions 1), 2) if, for some $q_0 \in Q$, the operator $A(q_0)$ has a bounded inverse.

4. Without dwelling on the characteristic property of the function of least deviation, since, obviously, theorem 2 from (1) carries over without change also to the operator-functions considered in the present work, we pass to the consideration of the question of uniqueness of the least-deviating function.

Theorem 2. *Let $A(q)$ satisfy the conditions of Theorem 1 and* $\dim H_1 < \dim H_2$. *Then, in order that for every function $f(q)$, continuous on Q and with values in H_2, there exist a unique function $A(q)x_0$ least deviating from it on Q, it is necessary and sufficient that, for $x \neq \theta_1$, the equation $A(q)x = \theta_2$ have not a single root on Q.*

The following theorem is a transfer to the case of operator-functions of the corresponding theorems from (4,5).

Theorem 3. *Let $A(q)$ be, for every $q \in Q$, a linear operator acting from H_1 into H_2, and let, for every fixed $x \in H_1$, the function $A(q)x$ be continuous on Q. Let the number n $(1 \le n < \infty)$ be such that*

$$(n-1)\dim H_2 < \dim H_1 \le n \dim H_2.$$

Then, for uniqueness of the function of least deviation, it is necessary and sufficient that the following conditions be fulfilled:

1) for $x \neq \theta_1$, the equation $A(q)x = \theta_2$ has on Q no more than $n-1$ roots;

2) for any distinct $q_1, q_2, \ldots, q_{n-1}$ from Q and any $y_1, y_2, \ldots, y_{n-1}$ from H_2, there exists a vector $x \in H_1$ such that

$$A(q_1)x = y_1, \quad A(q_2)x = y_2, \ldots, \quad A(q_{n-1})x = y_{n-1}.$$

When $\dim H_1 = n \dim H_2$, condition 2) is a consequence of condition 1).

5. In the case of Banach spaces we note the following theorems:

Theorem 4. Let $A(q)$, for each $q \in Q$, be a linear discontinuous operator acting from the Banach space B_1, with weakly compact sphere (for example, reflexive), into the Banach space B_2, and suppose that for each fixed $x \in B_1$ the function $A(q)x$ is continuous. Then, in order that for every continuous function $f(q)$ on Q with values in B_2 there exist a function $A(q)x_0$ of least deviation from it, it is necessary and sufficient that $A(q)$ possess property (a)

$$\max_{q \in Q} \|A(q)x\|_{B_2} \geq m\|x\|_{B_1} \quad \text{for all } x \in B_1.$$

Theorem 5. Let $A(q)$ satisfy the conditions of Theorem 4, suppose that B_2 is strictly convex with weakly compact sphere, and, in addition, has the following property: if, for some $q_1 \in Q$, the equation $A(q_1)x = \theta_2$ has a solution $x_0 \neq \theta_1$, then the adjoint equation $A^*(q_1)Y = \theta_1^*$ also has a solution $Y_0 \neq \theta_2^*$. Then, for uniqueness of the function of least deviation, it is necessary and sufficient that the equation $A(q)x = \theta_2$, for $x \neq \theta_1$, have no roots on Q.

The conditions of the last two theorems are satisfied, for example, by the operator-function $A(q) = E - T(q)$, where $T(q)$, for each $q \in Q$, is a completely continuous operator from B_1 into B_2.

Remarks. 1°. Theorem 3 also carries over to the case of Banach spaces B_1 and B_2 under the condition that B_2 is strictly convex, and Theorem 2—if, in addition, the sphere in B_2 is weakly compact.

2°. Let us note that the preceding theorems for the case of Banach spaces were formulated under the assumption that

$$R = \{x : \ A(q)x = \theta_2 \text{ for all } q \in Q\} = \theta_1.$$

In the contrary case, instead of B_1 one should consider the factor space B_1/R of elements X with norm

$$\|X\| = \inf_{x \in X} \|x\|_{B_1}.$$

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Received
24 IV 1957

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Note: Figure translations are in progress. See original paper for figures.

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