A. A. KOZMANOVA

1957

SovietRxiv

Abstract

Full Text

A. A. KOZMANOVA

PÓLYA' S THEOREM FOR ENTIRE FUNCTIONS OF TWO COMPLEX VARIABLES

(Presented by Academician M. A. Lavrent′ev, 19 XI 1956)

Let an entire function of exponential type of two complex variables p_1 and p_3 be given:

$$F(p_1, p_3) = \sum_{n=0}^{\infty} \sum_{m=0}^{n} a_{nm} p_3^{n-m} p_1^m .$$

A vector $\mathbf{p}(p_1, p_2, p_3)$ is called isotropic if $p_1^2 + p_2^2 + p_3^2 = 0$. We associate with $F(p_1, p_3)$ an entire function of the isotropic vector:

$$F(\mathbf{p}) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} a_{nm} p_3^{n-m} (p_1 + ip_2)^m + \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} a_{nm}^* p_3^{n-m} (p_1 - ip_2)^m . \quad (1)$$

The harmonic function

$$f(x, y, z) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} (-1)^m (n - |m|)! a_{nm} \frac{P_n^{|m|}(\cos Q) e^{im\varphi}}{r^{n+1}}, \quad a_{n(-m)} = a_{nm}^* . \quad (2)$$

will be called the function associated with $F(\mathbf{p})$.

An isotropic vector $\mathbf{p}$ can be represented as follows: $\mathbf{p} = \mathbf{p}' + i\mathbf{p}''$, where $\mathbf{p}'$ and $\mathbf{p}''$ are real vectors, $i = \sqrt{-1}$. From the isotropy of $\mathbf{p}$ it follows that $|\mathbf{p}'| = |\mathbf{p}''|$, $\mathbf{p}' \perp \mathbf{p}''$. Consequently, with each isotropic vector one may associate an orthogonal trihedron $OX'Y'Z'$ with center at O, directing the axis OZ' along the vector $\mathbf{p}'$, the axis OX' along the vector $\mathbf{p}''$, and giving $OX'Y'Z'$ the same orientation as the trihedron $OXYZ$. The trihedron $OX'Y'Z'$ can be characterized by means of the rotation of space $g(\varphi_1, \theta, \varphi_2)$, which carries the principal trihedron $OXYZ$ into $OX'Y'Z'$. Thus the isotropic vector $\mathbf{p}$ is determined by the positive number $\rho = |\mathbf{p}'| = |\mathbf{p}''|$ and the angles $\varphi_1, \theta, \varphi_2$. We note that

$$\mathbf{p}' = \rho(\sin\theta \cos\varphi \, \mathbf{i} + \sin\theta \sin\varphi \, \mathbf{j} + \cos\theta \, \mathbf{k}),$$

where $\varphi = \varphi_1 - \pi/2$ (^{1}).

The growth indicatrix of the entire function of exponential type $F(\mathbf{p})$ will be the function

$$h(\varphi_1, \theta, \varphi_2) = \lim_{\rho\to\infty} \frac{\ln |F(\mathbf{p})|}{\rho},$$

where $\mathbf{p}$ is a function of the variables $\rho, \varphi_1, \theta, \varphi_2$.

Let D be the convex hull of the singularities of the function $f(x, y, z)$. The supporting function of the domain D is the function

$$K(\varphi, \theta) = \max_{(x,y,z)\in D} \{x \sin\theta \cos\varphi + y \sin\theta \sin\varphi + z \cos\theta\},$$

where θ and φ are the angles of the spherical coordinate system.

Theorem 1. If

$$F(\mathbf{p}) = \sum_{n=0}^{\infty} \sum_{m=0}^{n} a_{nm} p_3^{n-m} (p_1 + ip_2)^m + \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} a^*_{nm} p_3^{n-m} (p_1 - p_2)^m.$$

an entire function of exponential type of the isotropic vector $\mathbf{p}$, then its indicatrix $h(\varphi_1, \theta, \varphi_2)$ is connected with the supporting function of the bounded convex hull of the singularities of the harmonic function associated with $F(\mathbf{p})$

$$f(x, y, z) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} a_{nm} \frac{(-1)^m}{(n - |m|)!} \frac{P_n^{|m|}(\cos\theta) e^{im\varphi}}{r^{n+1}}, \qquad a_{n(-m)} = a^*_{nm},$$

by the relation

$$\sup_{\varphi_2} h\left(\frac{\pi}{2} + \varphi; \theta, \varphi_2\right) = K(\varphi, \theta). \tag{3}$$

Let us prove that, for the functions $F(\mathbf{p})$ and $f(x, y, z)$, the following inversion formulas hold:

$$\frac{1}{4\pi} \mathbf{p} F(\mathbf{p}) = \iint_\sigma [\operatorname{grad} f \, \mathbf{n} \mathbf{p} e^{(\mathbf{pr})}] \, d\sigma, \tag{4}$$

$$2\pi f(x, y, z) = \iint_s F(\mathbf{p}) \frac{e^{-(\mathbf{pr})}}{\rho} \, dS, \tag{5}$$

where σ is a piecewise-smooth surface enclosing all singularities of $f(x, y, z)$; $\mathbf{p} = p_1\mathbf{i} + p_2\mathbf{j} + p_3\mathbf{k}$ is an isotropic vector; $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ is the radius vector of the point $M(x, y, z)$; $\mathbf{n}$ is the unit vector of the outward normal to σ; under the integral sign in (4) stands the triple product of vector functions defined in (1)*; s is the plane perpendicular to the vector $\mathbf{p}'(\rho, \varphi_1', \theta')$; φ_1' and θ' are arbitrarily fixed angles. In the plane s lies the vector $\mathbf{p}''(\rho, \varphi_1', \theta', \varphi_2)$ as φ_2 varies on the interval $[0, 2\pi]$ and $\rho \in [0, \infty)$.

We shall first show the validity of relation (4). We shall call a vector function $\vec{\varphi}(\mathbf{r})$ **potential-harmonic** if $\operatorname{div}\vec{\varphi} = 0$ and $\operatorname{rot}\vec{\varphi} = 0$ (^{2}). If $\vec{\varphi}$ and $\mathbf{g}$ are functions potential-harmonic in a domain containing $T + \sigma$, where σ is a piecewise-smooth boundary of the domain T, then

$$\iint_\sigma [\vec{\varphi}\,\mathbf{n}\mathbf{g}]\, d\sigma = 0 \quad (1).$$

It follows from this: let $\vec{\varphi}$ be a potential-harmonic function in the domain T with piecewise-smooth boundary σ; let r be the distance between the points $M_1(x, y, z)$ and $M(\xi, \eta, \zeta)$; then

$$\iint_\sigma \left[\operatorname{grad}\frac{1}{r}\,\mathbf{n}\vec{\varphi}\right] d\sigma = \begin{cases} \vec{\varphi}(x, y, z), & (x, y, z) \in T; \\ 0, & (x, y, z) \in T', \end{cases} \tag{6}$$

where T' is the domain lying outside T.

In a somewhat different form formula (6) was given by A. V. Bitsadze (^{2}). We next use the formula (^{3})

$$\frac{\partial^{n-m}}{\partial z^{n-m}}\left(\frac{\partial}{\partial x} \pm i\frac{\partial}{\partial y}\right)^m \frac{1}{r} = (-1)^{n-m}\frac{(n-m)!}{r^{n+1}} P_n^m(\cos\theta)e^{\pm im\varphi} \tag{7}$$

and the fact that series (2) admits, for sufficiently large r, termwise differentiation with respect to x, y, z and then termwise integration over the surface σ (the latter will be justified later).

The proof of formula (5) is based on the fact that, taking as s the plane $XOY - s'$, we obtain:

$$\iint_{s'} \frac{e^{-(\mathbf{pr})}}{\rho}\, dS = \int_0^\infty d\rho \int_0^{2\pi} e^{-\rho z - i\rho(x\cos\psi + y\sin\psi)}\, d\psi = \frac{2\pi}{r}. \tag{8}$$

* $[\mathbf{abc}] = -(\mathbf{bc})\,\mathbf{a} + (\mathbf{ca})\,\mathbf{b} - (\mathbf{ab})\,\mathbf{c}$.

Performing the rotation, we obtain that, in the general case,

$$\frac{2\pi}{r} = \iint_S \frac{e^{-(\rho r)}}{\rho}\, dS,$$

where S is the plane mentioned above. Then from (1) and (7), and from the possibility of termwise integration for sufficiently large r, upon substituting series (1) into integral (5), formula (5) follows. Along the way we obtain that series (2), for sufficiently large r, may be differentiated term by term with respect to x, y, z any number of times and integrated term by term over the surface σ. From formulas (5) and (6), (3) is obtained by arguments analogous to those used in proving Pólya' s theorem from the theory of entire functions [4].

Corollary 1. Theorem 1 remains valid if, as $f(x, y, z)$, one takes any function regular at infinity and harmonic outside some surface enclosing the origin.

Suppose that all singularities of the function $f(x, y, z)$ lie in the half-space $z < a$ $(a > 0)$. Denote by H the distance from the plane $z = a$ to the set of these singularities. The equality holds

$$H = a - \sup_{\varphi_2} h\left(\frac{\pi}{2}, 0, \varphi_2\right), \tag{9}$$

where h is the indicator of growth of the entire function $F(p)$ associated with $f(x, y, z)$.

The analogous question for the two-dimensional case was considered by A. Steiner [5].

Corollary 2. For the function $f(x, y, z)$ appearing in Corollary 1, the following is valid:

$$f(x, y, z) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} a_{nm} \frac{P_n^{|m|}(\cos\theta) e^{im\varphi}}{r^{n+1}}, \qquad a_{n(-m)} = a^*_{nm}, \tag{2'}$$

where the series on the right converges outside the sphere of radius r_0 passing through the singular point of the function $f(x, y, z)$ farthest from the origin, and, evidently,

$$r_0 = \sup_{\varphi,\theta} K(\varphi, \theta) = \sup_{\theta,\varphi,\varphi_2} h\left(\frac{\pi}{2} + \varphi, \theta, \varphi_2\right). \tag{10}$$

The expression on the right in (10) is the type of the function $F(p)$. The relation obtained coincides, to a certain extent, with the analogous fact from the theory of entire functions.

Corollary 3. Formula (5) may serve as a method for summing series $(2')$, since the integral in (5) converges outside the domain defined by the relation

$$K(\varphi, \theta) = \sup_{\varphi_2} h\left(\frac{\pi}{2} + \varphi, \theta, \varphi_2\right). \tag{11}$$

Corollary 4. From (7) and (8) we obtain

$$P_n^m(\cos\theta) e^{im\varphi} = \frac{(-i)^m}{2\pi} \frac{n!}{(n-m)!} \int_0^{2\pi} \frac{e^{im\psi}\, d\psi}{[\cos\theta + i\sin\theta\cos(\psi - \varphi)]^{n+1}}, \tag{12}$$

where $0 \leq \theta < \pi/2$.

Ural State University
named after A. M. Gorky

Received
11 IV 1956

REFERENCES

[1] V. K. Ivanov, *Investigations on the inverse problem of potential*, Dissertation, V. A. Steklov Mathematical Institute, Academy of Sciences of the USSR, 1955.
[2] A. V. Bitsadze, *Izv. Akad. Nauk SSSR*, ser. mat., **17**, 525 (1953).
[3] E. V. Hobson, *The Theory of Spherical and Ellipsoidal Functions*, Moscow, 1952.
[4] G. Polya, *Math. Zs.*, **29**, 549 (1929).
[5] A. Steiner, *Rend. Circolo Mat. Palermo*, **3**, 198 (1954).

Note: Figure translations are in progress. See original paper for figures.