MATHEMATICS

V. B. LIDSKII

1957

SovietRxiv

Abstract

Full Text

MATHEMATICS

V. B. LIDSKII

ON THE COMPLETENESS OF THE SYSTEM OF EIGEN AND ASSOCIATED ELEMENTS OF A COMPLETELY CONTINUOUS OPERATOR

(Presented by Academician M. V. Keldysh, 3 February 1957)

We consider a completely continuous operator T acting in a Hilbert space $\mathfrak{H}$. The operator is written in the form

$$T = A + iB, \tag{1}$$

where A and B are self-adjoint operators: $A = \frac{1}{2}(T + T^*)$, $B = \frac{1}{2i}(T - T^*)$. Below we prove a theorem containing conditions under which the system of eigen and associated elements of the operator T is complete. In the proof, following M. V. Keldysh ([1]), we apply the Phragmén–Lindelöf theorem in order to estimate an entire function arising under the assumption that the system is not complete (see also ([3])). Here a very essential role is played by an estimate of the growth of entire functions, which follows from results of M. S. Livshits ([2]).

Theorem. *Let the self-adjoint operators A and B be semidefinite* and let the operator B have a finite trace

$$\operatorname{Sp} B = \sum_{s=1}^{\infty} |\mu_s| < \infty \tag{2}$$

(μ_s are the eigenvalues of B).

Then the system of eigen and associated elements of the operator $T = A + iB$, corresponding to the nonzero points of the spectrum, is complete in the range of values of the operator T.

Remark. If to the indicated system one adds the system of eigen elements corresponding to the zero eigenvalue, then one obtains a system complete in the whole Hilbert space $\mathfrak{H}$.

Proof. Denote by Q_1 the closed subspace spanned by the eigen and associated elements $\{f_s\}$ of the operator T that correspond to the nonzero points of the spectrum. Let Q_2 be the orthogonal complement of Q_1. It is easy to prove that the operator T^*, adjoint to T, is invariant on the subspace Q_2 and, considered on this subspace, has only one point of the spectrum—zero.

In view of this last remark, the function

$$\varphi(\zeta) = \zeta\,((T^* - \zeta E)^{-1}h,\ g), \tag{3}$$

where h and g are two as yet fixed elements from Q_2, is regular in the whole ζ-plane except for the point $\zeta = 0$. It is easy to verify that $\varphi(\zeta)$ is regular at $\zeta = \infty$, by expanding the resolvent $(T^* - \zeta E)^{-1}$ in a series in powers of ζ^{-1}:

$$\varphi(\zeta) = -(h,g) - (T^*h,g)\zeta^{-1} - (T^{*2}h,g)\zeta^{-2} - \cdots. \tag{4}$$

* In other words, the quadratic forms (Af,f) and (Bf,f) do not change sign.

We shall prove that

$$\varphi(\zeta) = \text{const}. \tag{5}$$

For this, let us first note that, by virtue of condition (2) of the theorem being proved, the operator T^* considered on Q_2 is an operator of class $i\Omega$ $\left({}^{(2)}, \text{p. } 145\right)$, and, as follows from $\left({}^{2}\right)$ (pp. 161, 186), for its resolvent $(T^* - \zeta E)^{-1}$ in a neighborhood of the isolated point of the spectrum $\zeta = 0$, and hence also for the function $\varphi(\zeta)$ introduced by us, the inequality

$$|\varphi(\zeta)| \le e^{a|\zeta|} \tag{6}$$

holds.

Let us further assume, for definiteness, that the operators A and B in formula (1) are nonnegative. We shall prove that in this case the function $\varphi(\zeta)$ remains bounded if $\zeta \to 0$ along any ray not lying in the fourth quadrant. To this end put

$$(T^* - \zeta E)^{-1}h = e. \tag{7}$$

Applying the operator $T^* - \zeta E$ to both sides of this relation, we obtain $(T^* - \zeta E)e = h$, or, more explicitly: $Ae - iBe - \zeta e = h$. Multiplying this equality scalarly by e, and separating the real and imaginary parts, we shall have as a result:

$$(Ae,e) - (e,e)\operatorname{Re}\zeta = \operatorname{Re}(h,e); \tag{8}$$

$$(Be, e) + (e, e)\operatorname{Im}\zeta = -\operatorname{Im}(h, e). \tag{9}$$

If now $\zeta \to 0$ along a ray lying in the upper half-plane, then in formula (9) $\operatorname{Im}\zeta > 0$, $(Be, e) \geq 0$, and with the aid of the Cauchy—Bunyakovsky inequality we easily obtain:

$$\|e\| \leq \|h\|(\operatorname{Im}\zeta)^{-1}, \tag{10}$$

whence for the functions $\varphi(\zeta)$ we shall have:

$$|\varphi(\zeta)| \leq |\zeta|\|e\|\|g\| \leq |\zeta|(\operatorname{Im}\zeta)^{-1}\|h\|\|g\| = C. \tag{11}$$

If, however, $\zeta \to 0$ along a ray from the left half-plane, then, using formula (8), in an analogous way we obtain

$$|\varphi(\zeta)| \leq |\zeta|\,|\operatorname{Re}\zeta|^{-1}\|h\|\|g\| = C'. \tag{12}$$

Inequalities (6), (11), and (12) make it possible to apply the Phragmén—Lindelöf theorem to the function $\varphi(\zeta)$ as $\zeta \to 0$ and to conclude that $\varphi(\zeta)$ is bounded in a neighborhood of zero, whence it follows that $\varphi(\zeta)$ is identically equal to a constant.

From the expansion (4) it then follows that

$$(T^*h, g) = 0, \tag{13}$$

whence

$$(h, Tg) = 0. \tag{14}$$

But h and g are arbitrary elements of Q_2. Therefore from formula (14) we easily conclude that $TQ_2 \subset Q_1$. Since, moreover, $TQ_1 \subset Q_1$ and $\mathfrak{H} = Q_1 \oplus Q_2$, it follows that $Tf \in Q_1$ for any $f \in \mathfrak{H}$. The theorem is proved.

In conclusion let us establish the validity of the remark to the theorem. Suppose that for some element h, $T^*h = 0$; then $h \in Q_2$. Indeed, h in

in this case is orthogonal to any eigen and associated element of the system $\{f_s\}$ of the operator T, and consequently also to every element of Q_1. On the other hand, from formula (13) it follows, in view of the arbitrariness of h and g and the invariance of T^* on Q_2, that for any $h \in Q_2$, $T^*h = 0$. Thus, Q_2 consists of those and only those elements for which

$$T^*h = 0. \tag{15}$$

But from formula (15) it follows that $(Ah, h) - i(Bh, h) = 0$, and hence, in view of the sign-definiteness of the operators A and B, simultaneously $Ah = 0$ and $Bh = 0$, whence we conclude that equality (15) entails

$$Th = 0. \tag{16}$$

The equalities (15) and (16) are, obviously, equivalent. This completes the proof.

Moscow
Institute of Physics and Technology

Received
29 I 1957

CITED LITERATURE

[1] M. V. Keldysh, DAN, **77**, No. 1 (1951).
[2] M. S. Livshits, Matem. sborn., **34** (**76**), 1 (1954).
[3] F. Browder, Proc. Nat. Acad. Sci., **39**, No. 5 (1953).

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.