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Abstract

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THE METHOD OF ORTHOGONAL PROJECTIONS FOR A FINITE-DIFFERENCE ANALOGUE OF A CERTAIN SYSTEM OF EQUATIONS

(Presented by Academician S. L. Sobolev on 16 XI 1956)

In the present note we investigate properties of the solutions of a finite-difference analogue of the system of equations:

$$\frac{\partial \mathbf{U}}{\partial t} = A\mathbf{U} - \operatorname{grad} p + \mathbf{F}, \qquad \operatorname{div} \mathbf{U} = 0, \tag{1}$$

where $\mathbf{U} = (u_1(x_1, x_2, x_3, t), u_2(x_1, x_2, x_3, t), u_3(x_1, x_2, x_3, t))$; $\mathbf{F} = (f_1, f_2, f_3)$; A is a matrix with bounded elements.

For system (1) one poses either the Cauchy problem, in which case one is given

$$\mathbf{U}\big|_{t=0} = \mathbf{U}_0(x_1, x_2, x_3), \tag{2}$$

or a mixed problem: one seeks a solution of (1) in a simply connected domain Ω, satisfying condition (2) and one more of the two conditions: either

$$p\big|_S = 0, \tag{3}$$

or

$$\sum_{i=1}^{3} u_i \cos(n, x_i)\big|_S = U_n\big|_S = 0, \tag{4}$$

where S is the boundary of the domain Ω, and n is the normal to the boundary. A system of type (1) was investigated by S. L. Sobolev (^{1}); the proofs of existence of solutions of the listed problems for system (1) do not differ in principle from the proofs in (^{1}).

Let the space $R_3(x_1, x_2, x_3)$ be given. The set of points $x \in R_3$ with coordinates $x_i = k_i h$, $i = 1, 2, 3$, where $h > 0$ and k_i are integers, will be denoted by M_h.

The set of points $x \in M_h$ for which $\sum_{i=1}^{3} k_i = 2j,\ j = 0, \pm 1, \pm 2, \ldots$, will be denoted by M_{1h}, and the set of points $x \in M_h$ for which $\sum_{i=1}^{3} k_i = 2j + 1$ will be denoted by M_{2h}. If the domain Ω is finite, then we shall consider that $x(k_1 h, k_2 h, k_3 h) \in \Omega_{2h}$ if $x \in M_{2h}$ and the octahedron with center at the point x and with diagonals parallel to the coordinate axes and of length $4h$ belongs to $\overline{\Omega}$. Define the boundary points for Ω_{2h}: we shall say that $x \in S_{2h}$ if at distance $2h$ from the point x there are points of M_{2h} both belonging and not belonging to Ω_{2h}; denote $\overline{\Omega}_{2h} = \Omega_{2h} + S_{2h}$. We shall say that $x \in \Omega_{1h}$ if $x \in M_{1h}$ and at distance h from the point x there are 6 points of the set $\overline{\Omega}_{2h}$. Let a function $\varphi(x_1, x_2, x_3)$ be given on M_{2h}; then denote:

$$\varphi_{x_1}(x_1, x_2, x_3) = \frac{1}{2h}(\varphi(x_1 + h, x_2, x_3) - \varphi(x_1 - h, x_2, x_3)),$$

where $x(x_1, x_2, x_3) \in M_{1h}$; $\varphi_{x_2}, \varphi_{x_3}$ are defined analogously, and φ_{x_i}, $i = 1, 2, 3$, are regarded as defined at the points $x \in M_{1h}$; φ_{x_i} are defined analogously at the points $x \in M_{2h}$ through the values of the function φ prescribed on the set M_{1h}.

Let H_{ih}, $i = 1, 2$, be the spaces of vectors $\mathbf{v}$ defined in Ω_{ih}, such that $h^3 \sum_{\Omega_{ih}} \|\mathbf{v}\|^2 < C$. We introduce scalar products in H_{ih}:

$$(\mathbf{v}^{(1)}, \mathbf{v}^{(2)})_{ih} = h^3 \sum_{\Omega_{ih}} \left(v_1^{(1)} v_1^{(2)} + v_2^{(1)} v_2^{(2)} + v_3^{(1)} v_3^{(2)} \right); \tag{5}$$

the notions of difference gradient, difference curl, and difference divergence:

$$\operatorname{grad}_h \varphi = (\varphi_{x_1}, \varphi_{x_2}, \varphi_{x_3}), \tag{6}$$

$$\operatorname{rot}_h \vec{\psi} = ((\psi_{3x_2} - \psi_{2x_3}), (\psi_{1x_3} - \psi_{3x_1}), (\psi_{2x_1} - \psi_{1x_2})), \tag{7}$$

$$\operatorname{div}_h \mathbf{v} = v_{1x_1} + v_{2x_2} + v_{3x_3}. \tag{8}$$

Theorem 1. For a vector $\mathbf{v}$, defined on M_{1h}, to be representable in the form $\mathbf{v} = \operatorname{grad}_h \varphi$, where φ is defined on M_{2h}, it is necessary and sufficient that

$$\operatorname{rot}_h \mathbf{v} = 0.$$

Theorem 2. For a vector $\mathbf{v}$, defined on M_{1h}, to be representable in the form $\mathbf{v} = \operatorname{rot}_h \vec{\psi}$, where the vector $\vec{\psi}$ is defined on M_{2h}, it is necessary and sufficient that

$$\operatorname{div}_h \mathbf{v} = 0.$$

The method of proof of Theorems 1 and 2 is the same as in the continuous case (^{2}).

In H_{1h} lie the linear manifold G_{1h} of vectors of the form $\mathbf{v}_1 = \operatorname{grad}_h \varphi$ and the linear manifold J_{1h} of vectors of the form $\mathbf{v}_2 = \operatorname{rot}_h \vec{\psi}$. We also introduce the linear manifolds of vectors H^0_{1h}, G_{0h}, J_{0h}. We assume that: $\mathbf{v} \in H^0_{1h}$ if $\mathbf{v} \in H_{1h}$ and $\mathbf{v} \equiv 0$ outside $\Omega_{vh} \subseteq \Omega_{1h}$; $\mathbf{v}_1 \in G_{0h}$ if $\mathbf{v}_1 = \operatorname{grad}_h \varphi \in G_{1h}$ and $\varphi \equiv 0$ outside $\Omega_{v_1 h} \subseteq \Omega_{2h}$; $\mathbf{v}_2 \in J_{0h}$, if $\mathbf{v}_2 = \operatorname{rot}_h \vec{\psi} \in J_{1h}$ and $\vec{\psi} \equiv 0$ outside $\Omega_{v_2 h} \subseteq \Omega_{2h}$.

Lemma 1. In the case when Ω is the whole space, an element $\mathbf{v}$ of H_{1h}, orthogonal to all elements of G_{0h} and J_{0h}, can only be identically zero.

Indeed, since $\mathbf{v} \perp J_{0h}$ and G_{0h}, then $\mathbf{v} \perp \Delta_{2h}\mathbf{w}$, if $\mathbf{w} \in H^0_{1h}$, because

$$\Delta_{2h}\mathbf{w} = \operatorname{grad}_h \operatorname{div}_h \mathbf{w} - \operatorname{rot}_h \operatorname{rot}_h \mathbf{w},$$

and then $\Delta_{2h} v_i = 0$, i.e. $\mathbf{v} \equiv 0$, since $h^3 \sum_{M_{1h}} \|v\|^2 < C$.

Lemma 2. The manifold G_{0h} is orthogonal to the manifold J_{1h}.

Lemma 3. The manifold G_{1h} is orthogonal to the manifold J_{0h}.

Corollary. The manifold J_{0h} is orthogonal to the manifold G_{0h}.

Theorem 3. In the case when Ω is the whole space, the space H_{1h} can be represented in the form

$$H_{1h} = J_h \oplus G_h,$$

where $J_h = \overline{J_{0h}} = \overline{J_{1h}}$, $G_h = \overline{G_{0h}} = \overline{G_{1h}}$ (the bar over a letter denotes the closure of the corresponding space).

For finite domains the following results hold.

Lemma 4. Every vector $\mathbf{v}$ from H_{1h}, orthogonal to J_{0h} and G_{0h} simultaneously, is a harmonic vector, i.e. its curl, its divergence, and $\Delta_{2h}\mathbf{v}$ are equal to zero.

Let $\mathbf{v}_1 = \operatorname{grad}_h \varphi_1 \in G_{0h}$; then

$$(\mathbf{v}, \mathbf{v}_1)_{1h} = 0 = (\mathbf{v}, \operatorname{grad}_h \varphi_1)_{1h} = -(\varphi_1, \operatorname{div}_h \mathbf{v})_{2h},$$

i.e. $\operatorname{div}_h \mathbf{v} = 0$ in Ω_{2h}.

Let $\mathbf{v}_2 = \operatorname{rot}_h \vec{\psi} \in J_{0h}$; then

$$(\mathbf{v}, \mathbf{v}_2)_{1h} = 0 = (\mathbf{v}, \operatorname{rot}_h \vec{\psi})_{1h} = (\vec{\psi}, \operatorname{rot}_h \mathbf{v})_{2h},$$

i.e. $\mathrm{rot}_h\,\mathbf{v} = 0$ in Ω_{2h}.

The fact that $\Delta_{2h}\mathbf{v} = 0$ follows from the formula for $\Delta_{2h}\mathbf{v}$.

Lemma 5. A vector $\mathbf{v}$ orthogonal to G_{0h} and J_{1h} simultaneously is identically zero.

Indeed, since $\mathbf{v} \perp G_{0h}$, $\mathrm{div}_h\,\mathbf{v} = 0$, i.e. $\mathbf{v} \in J_{1h}$, and since $\mathbf{v} \perp J_{1h}$, it follows that $\mathbf{v} = 0$.

Lemma 6. A vector $\mathbf{v}$ orthogonal to G_{1h} and J_{0h} simultaneously is identically zero.

Indeed, since $\mathbf{v} \perp J_{0h}$, $\mathrm{rot}_h\,\mathbf{v} = 0$, i.e. $\mathbf{v} \in G_{1h}$, and since $\mathbf{v} \perp G_{1h}$, it follows that $\mathbf{v} = 0$.

Theorem 4. The space H_{1h} admits the representation

$$H_{1h} = G_{0h} \oplus I_h \oplus J_{0h},$$

where $I_h = G_{1h}J_{1h}$.

We shall now construct a difference analogue of system (1). In the space $R_4(x_1, x_2, x_3, t)$ consider the set of points (x, t) with coordinates $x_i = k_i h,\ t = k_0\Delta t,\ i = 1, 2, 3,\ \Delta t > 0$; denote the set of points (x, t) such that $x \in \Omega_{ih}$ by D_{ih}. Put

$$\mathbf{U}_t(x_1, x_2, x_3, t) = \frac{1}{\Delta t}(\mathbf{U}(x_1, x_2, x_3, t) - \mathbf{U}(x_1, x_2, x_3, t - \Delta t)),$$

$$\mathbf{U}_{\mathrm{cp}}(x_1, x_2, x_3, t) = \alpha\mathbf{U}(x_1, x_2, x_3, t) + \beta\mathbf{U}(x_1, x_2, x_3, t - \Delta t),$$

where $\alpha \geq 0,\ \beta \geq 0$, and $\alpha + \beta = 1$.

At the points $(x, t) \in D_{1h}$ replace system (1) by the equations

$$\mathbf{U}_t = A\mathbf{U}_{\mathrm{cp}} - \mathrm{grad}\,p + \mathbf{F} \quad \text{and} \quad \mathrm{div}_h\,\mathbf{U} = 0 \tag{9}$$

at the points $(x, t) \in D_{2h}$. We shall prove the existence of a solution of (9).

I. Condition (4) for smooth functions and a smooth boundary of the domain Ω is equivalent to condition (1)

$$\int_\Omega (\mathbf{U}, \mathrm{grad}\,\varphi)\,d\Omega = 0 \quad \text{for any } \varphi. \tag{10}$$

For system (9), replace condition (10) by the condition that

$$(\mathbf{U}, \mathrm{grad}_h\,\varphi)_{1h} = 0 \quad \text{for any } \varphi. \tag{11}$$

Consider the solution of (9), replacing (4) by the requirement that $\mathbf{U}$ be an arbitrary element of J_{0h}. Then $\mathbf{v}_1 = \mathrm{grad}_h\,p$ is determined by the formula

$$\mathbf{v}_1 = P^*_{0h}\{A\mathbf{U}_{\mathrm{cp}} + \mathbf{F}\}, \tag{12}$$

and

$$\mathbf{U}_t = P_{0h}\{A\mathbf{U}_{\text{cp}} + \mathbf{F}\}, \tag{13}$$

where P^*_{0h}, P_{0h} are the projection operators of H_{1h}, respectively onto G_{1h} and J_{0h}.

II. Condition (3) for a smooth function p and a smooth boundary of the domain Ω is equivalent to condition (1)

$$\int_\Omega (\mathbf{U}, \operatorname{grad} p)\, d\Omega = 0 \quad \text{for any } \mathbf{U} \in J_1. \tag{14}$$

For system (9), replace (14) by the condition that

$$(\mathbf{U}, \operatorname{grad}_h p)_{1h} = 0 \quad \text{for any } \mathbf{U} \in J_{1h}. \tag{15}$$

Then

$$\mathbf{v}_1 = \operatorname{grad}_h p = P^*_{1h}\{A\mathbf{U}_{\text{cp}} + \mathbf{F}\}, \tag{16}$$

$$\mathbf{U}_t = P_{1h}\{A\mathbf{U}_{\text{cp}} + \mathbf{F}\}, \tag{17}$$

where P^*_{1h}, P_{1h} are the projection operators of H_{1h}, respectively, onto G_{0h} and J_{1h}.

III. Similarly, solutions of (9) are found for the Cauchy problem:

$$\mathbf{v}_1 = \operatorname{grad}_h p = P^*_{h}\{A\mathbf{U}_{\text{cp}} + \mathbf{F}\}, \tag{18}$$

$$\mathbf{U}_t = P_{h}\{A\mathbf{U}_{\text{cp}} + \mathbf{F}\}, \tag{19}$$

where P^*_h, P_h are the projection operators of H_{1h}, respectively, onto $\overline{G}_{0h}$ and $\overline{J}_{0h}$.

Let us find the solution of (13) in the form of a power series; for (17) and (19) the solutions are found analogously. If we denote

$$P_{0h}A\mathbf{v} = B_h\mathbf{v}, \qquad \Gamma_{h\Delta t} = \frac{1}{\Delta t}\ln\bigl[(E - B_h\alpha\Delta t)^{-1}(E + B_h\beta\Delta t)\bigr],$$

then the solution of the problem

$$\mathbf{v}_t = B_h\mathbf{v}_{\text{cp}}, \qquad \mathbf{v}\big|_{t=0} = \mathbf{v}_0 \quad \text{for } t_i = i\Delta t$$

is given by the formula

$$\mathbf{v}(t_i) = \exp[\Gamma_{h\Delta t} t_i]\mathbf{v}_0 = \sum_{n=0}^{\infty} \frac{t_i^n}{n!}(\Gamma_{h\Delta t})^n \mathbf{v}_0. \tag{20}$$

For the nonhomogeneous equation

$$\mathbf{U}_t = B_h \mathbf{U}_{\text{cp}} + P_{0h}\mathbf{F}$$

the solution is given by the formula

$$\mathbf{U}(t_i) = \exp[\Gamma_{h\Delta t} t_i]\mathbf{U}_0 + \Delta t \sum_{\tau_j=\Delta t}^{t_i} \exp[\Gamma_{h\Delta t}(t_i - \tau_j)](E - B_h \alpha \Delta t)^{-1} P_{0h}\mathbf{F}(\tau_j). \tag{21}$$

From the form of the formulas giving the explicit solution there follows the correctness of the solutions of the difference schemes considered, as well as the convergence in $W_2^{(1)}$ of the approximate solutions to the exact ones, provided $\mathbf{F}$ and $\mathbf{U}_0$ have the corresponding smoothness.

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Note: Figure translations are in progress. See original paper for figures.

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