On Completely Continuous Vector Fields in a Banach Space

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Abstract

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On Completely Continuous Vector Fields in a Banach Space

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In this note R is a real Banach space; Λ is the empty set; $E \subset R$; $E_i = (E)_i$ is the set of interior points of E; $E_g = (E)_g = \overline{E} \setminus E_i$ is the boundary of E; e is the identity mapping of R onto itself. A mapping f of a set E into R is called a **completely continuous vector field** defined in E if $e - f$ is a completely continuous mapping of E into R. A mapping f of a set E into R will be called a **local vector field** defined on E if for every point $x_0 \in E$ there exists its neighborhood $O^x = O^x(x_0)$ in R such that f is a completely continuous vector field defined on $O^x \cap E$. A mapping f of a set E into R will be called a **completely continuous vector field** defined inside E, if for every closed bounded $E_1 \subseteq E$ the mapping f is a completely continuous vector field defined on E_1. By $m(E)$ we denote the cardinality of the set E.

§ 1. In this paragraph G is an open subset of R; f is a local vector field defined in G; $x_0 \in G$, $y_0 = f(x_0)$; by $O^x = O^x(x_0)$, $O^y = O^y(y_0)$ we denote bounded neighborhoods of the points x_0, y_0 in R, and we always assume that $\overline{O^x} \subseteq G$. The point x_0 will be called an **isolated point** of the field f if $f(x_0 + h) \neq f(x_0)$ for all sufficiently small $h \neq 0$. If x_0 is an isolated point of the field f, then the local degree $\gamma(f, x_0)$ is defined. An isolated point x_0 of the field f will be called a **regular point** of the field f if $\gamma(f, x_0) \neq 0$. The point x_0 will be called a **point of regular differentiability** of the field f if f is differentiable at the point x_0 in the sense of Fréchet and $df(x_0, h) \neq 0$ for all $h \neq 0$. We denote the sets of isolated points, regular points, and points of regular differentiability of the field f, respectively, by G^i, G^r, G^{rd}. Finally, put

$$G^+ = \{x : x \in G^i,\ \gamma(f, x) > 0\}, \qquad G^- = \{x : x \in G^i,\ \gamma(f, x) < 0\},$$

$$G^0 = \{x : x \in G^i,\ \gamma(f, x) = 0\}, \qquad G^1 = \{x : x \in G^i,\ |\gamma(f, x)| = 1\}.$$

It is known or obvious that

$$G^{rd} \subseteq G^1 \subseteq G^r = G^+ \cup G^- \subseteq G^i = G^+ \cup G^- \cup G^0.$$

The point x_0 will be called a **point of local topologicity** of the field f if there exist $O^x = O^x(x_0)$, $O^y = O^y(y_0)$ such that f maps O^x topologically onto O^y. The point x_0 will be called a **point of openness** of the field f if for every $O^x = O^x(x_0)$ we have $y_0 \in (fO^x)_i$. The field f is called **locally topological** (**open**) in G if every point of G is a point of local topologicity (openness) of the field f.

It is obvious that if f is open in G, $O \subseteq G$, and O is open in R, then fO is open in R.

The following assertions hold:

1) Let $x_0 \in G^r$, $O^x = O^x(x_0)$. Then for every sufficiently small (connected) $O^y = O^y(y_0)$ there exists (one and only one connected) $O_1^x = O_1^x(x_0)$, contained in O^x, such that

$$\overline{O}_1^x \cap f^{-1}y_0 = \{x_0\}, \quad fO_1^x = O^y, \quad f(O_1^x)_g = (O^y)_g.$$

2) If $x_0 \in G^r$, then x_0 is a point of openness of the field f.

3) Let $G = G^r$. Then f is open in G, and the set of points of local topologicity of the field f is open in R and everywhere dense in G.

4) Let $G = G^{rd}$. Then f is locally topological in G, and $\gamma(f, x)$ is constant on each component of the set G.

5) Let, in addition, G be bounded, and let f be a completely continuous vector field given in $\overline{G}$. Then $f\overline{G} = \overline{fG}$, and the degree $\gamma(G, f, y)$ is defined and is constant on each component of the set $R \setminus fG_g$.

6) Let $R \neq R^1$, and let $O^x = O^x(x_0)$ be such that $(O^x \setminus \{x_0\}) \subseteq G^{rd}$. Then $x_0 \in G^r$, and for every sufficiently small connected $O^y = O^y(y_0)$ there exists one and only one connected $O_1^x = O_1^x(x_0) \subseteq O^x$ such that $fO_1^x = O^y$, $f(O_1^x)_g = (O^y)_g$, $m(O_1^x \cap f^{-1}y_0) = 1$, $m(O_1^x \cap f^{-1}y) = |\gamma(f, x_0)| = \text{const} > 0$ for all $y \in O^y \setminus \{y_0\}$.

7) Let $R \neq R^1$, and let $G \setminus G^{rd}$ be an isolated set. Then:

7,1) $G = G^r$, in particular (see 3)), f is open in G;

7,2) $\operatorname{sign} \gamma(f, x)$ is constant on each component of the set G;

7,3) the set G^1 coincides with the set of points of local topologicity of the field f, and hence (see 7,1), 3)), G^1 is open in R and everywhere dense in G;

In assertions 7,4)–7,10) we assume that f is a completely continuous vector field given inside G; $G \setminus G^{rd}$ is an isolated set.

7,4) the set $G \cap f^{-1}y$ is isolated and at most countable for any $y \in R$;

7,5) if $E = F_\sigma(R) \subseteq G$, $E_i = \Lambda$, then $fE = F_\sigma(R)$, $(fE)_i = \Lambda$;

7,6) $(f(G \cap (G \setminus G^{rd})))_i = \Lambda$;

7,7) if $E = F_\sigma(R) \subseteq G$, $E_i = \Lambda$, O is open in R and bounded, $O \subseteq G$, $O_g \subseteq E$, then the set $R \setminus E$ is disconnected;

7,8) if F is closed in R, $F \subseteq G$, $F_i = \Lambda$, and O is a bounded component of the set $R \setminus F$, $O \subseteq G$, then the set $R \setminus fF$ is disconnected;

7,9) if F is closed in R, $F \subseteq G$, $F_i = \Lambda$, and moreover the cardinality of the system of bounded components of the set $R \setminus F$ is greater than the cardinality of the system of bounded components of the set $R \setminus G$, then the set $R \setminus fF$ is disconnected;

7,10) if O is open in R, $\pi(O) < 1$, $\pi(G) < \tau$, then $\pi(G \cap f^{-1}O) < \tau$ (see below).

Remark 1. Let O be open in R. The indicated system $\{O_\alpha\}$ of pairwise disjoint bounded sets O_α open in R will be called a **system of holes in the set** O if $(O_\alpha)_g \subseteq O$, $O_\alpha \cap (R \setminus O) \neq \Lambda$ for every α. We shall write $\pi(O) < \tau$ if every system of holes in the set O has cardinality $< \tau$.

8) Let $R \neq R^1$ and, in addition, let G be bounded, and let f be a completely continuous vector field given in $\overline{G}$. Let, further, $G \setminus G^{rd}$ be an isolated set. Finally, let Γ be a component of the set $R \setminus fG_g$. Then: either $\Gamma \cap fG = \Lambda$, or $\Gamma \subset fG$. There exist an integer nonnegative number $\beta = \beta(\Gamma)$ and a set $B(\Gamma) \subseteq \Gamma$, open in R and everywhere dense in Γ, such that for any $y \in B(\Gamma)$ we have $m(G \cap f^{-1}y) = \beta(\Gamma)$.

If Γ is unbounded, then $\Gamma \cap fG = \Lambda$. If U is a component of the set $G \setminus f^{-1}fG_g$, then fU is a component of the set $R \setminus fG_g$.

§ 2. In this section X is a locally linearly connected topological space, $Z = [X, R]$ is the topological product of X by R; $x \in X$; $y, u \in R$; $O^x = O^x(x_0)$, $O^y = O^y(y_0)$, $O^u = O^u(u_0)$, $O^z = O^z(x_0, y_0)$ are absolute neighborhoods, respectively, of the points $x_0, y_0, u_0, (x_0, y_0)$ in the spaces X, R, R, Z; π_x, π_y are the projections of Z onto X, R, respectively; $G \subseteq Z$, G is open in Z, $(x_0, y_0) \in G$; $M \subseteq R$. We further set

$$M(x) = \{y : (x, y) \in M\}, \qquad \widetilde{M} = \{(x, y) : (x, y) \in \overline{M},\ x \in \pi_x M\}.$$

M is bounded with respect to y if $\pi_y M$ is bounded in R; M is connected with respect to y if $M(x)$ is connected for every $x \in \pi_x M$.

φ is a continuous mapping of G into R such that for every $(x_0, y_0) \in G$ there exists $O^z = O^z(x_0, y_0)$ such that φO^z is relatively compact in R; f is a mapping of G into R, given by the formula $f(x, y) = y - \varphi(x, y)$; f_x is a mapping of $G(x)$ into R, given by the formula $f_x(y) = f(x, y)$ $(y \in G(x))$.

It is easy to see that for every x the set $G(x)$ is open in R, and f_x is a completely continuous vector field defined in $G(x)$. Therefore we may set

$$G^r = \{(x, y) : y \in (G(x))^r\}, \qquad G^{rd} = \{(x, y) : y \in (G(x))^{rd}\},$$

$$G^+ = \{(x,y) : y \in (G(x))^+\}, \qquad G^- = \{(x,y) : y \in (G(x))^-\}.$$

Finally, we set $u_0 = f(x_0, y_0) = f_{x_0}(y_0)$.

The following assertions hold:

9) Let $(x_0, y_0) \in G^r$ and let $O^z = O^z(x_0, y_0)$ be given. Then for every sufficiently small connected $O^u = O^u(u_0)$ there is a neighborhood $O_1^z = O_1^z(x_0, y_0) \subseteq O^z$, connected with respect to y, bounded with respect to y, and connected, such that:

9,1) $O_1^z \subseteq G$;

9,2) $\overline{O_1^z}(x_0) \cap f_{x_0}^{-1} u_0 = \{y_0\}$;

9,3) $f_x O_1^z(x) = O^u$ for any $x \in \pi_x O_1^z$;

9,4) $f_x(O_1^z(x))_g = (O^u)_g$ for any $x \in \pi_x O_1^z$;

9,5) For any $x \in \pi_x O_1^z$, $u \in O^u$, the degree $\gamma(O_1^z(x), f_x, u)$ is defined and

$$\gamma(O_1^z(x), f_x, u) = \gamma(f_{x_0}, u_0).$$

10) Let $G = G^{rd}$. Then each of the sets G^+, G^- is open in Z. Hence it follows that $\gamma(f_x, y)$ is constant on each component of the set G.

11) Let $O^z = O^z(x_0, y_0) \subseteq G^{rd}$. Then for every sufficiently small connected $O^u = O^u(u_0)$ there is a neighborhood $O_1^z = O_1^z(x_0, y_0)$, connected, bounded with respect to y, and connected with respect to y, contained in O^z and such that the mapping ψ, defined on the set $[\pi_x O_1^z, O^u]$ by the formula

$$\psi(x, u) = O_1^z(x) \cap f_x^{-1} u \quad ((x, u) \in [\pi_x O_1^z, O^u])$$

is a single-valued continuous open mapping of the set $[\pi_x O_1^z, O^u]$ into the space R_y, and we have:

11,1) for every $x \in \pi_x O_1^z$, the mapping f_x is a topologically regularly differentiable mapping of $O_1^z(x)$ onto O^u, and the mapping ψ_x

$(\psi_x(u) = \psi(x, u))$ is a topological regularly differentiable mapping of O^z onto $O_1^z(x)$;

$11, 2)$ for every $(x, u) \in [\pi_x O_1^z, O^u]$;

$11, 2, 1)$ $(x, \psi(x, u)) \in O_1^z$;

$11, 2, 2)$ $f(x, \psi(x, u)) = u$.

12) Let Φ be a bicompact set $\subset G = G^{rd}$. Then

$$\sup_x \left(\sup_n m\left(\Phi(x) \cap f_x^{-1} u\right) \right) < +\infty.$$

Remark 2. In the special case $R = R^n$, part of the results of this note was obtained in papers ([1–10]), as a rule, under stronger (and sometimes under weaker) restrictions on the mappings and sets. Some of the results are essentially new even in the case $R = R^n$.

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Note: Figure translations are in progress. See original paper for figures.