ON THE DEFINITION OF GENERALIZED FUNCTIONS

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Abstract

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The concept of a generalized function was first introduced by S. L. Sobolev ([1,2]) as a linear functional in a space of "fundamental functions." At present there are a number of approaches to the definition of a generalized function (see, for example, ([3–9])). The aim of this note is, starting from the definition of Mikusiński–Korevaar ([5]), to give a general definition of various classes of generalized functions.

By a class of estimates $\mathfrak{M}$ we shall mean any collection of nonnegative functions (all functions under consideration are assumed to be real-valued and continuous on the axis $-\infty < x < \infty$) having the following properties:

1) if $0 \leq m(x) \leq m_1(x) \in \mathfrak{M}$, then $m(x) \in \mathfrak{M}$;
2) if $m_1(x) \in \mathfrak{M}$ and $m_2(x) \in \mathfrak{M}$, then $m_1(x) + m_2(x) \in \mathfrak{M}$;
3) $1 \in \mathfrak{M}$;
4) if $m(x) \in \mathfrak{M}$, then
$$\max_{0 \leq t \leq 1} m(tx) \in \mathfrak{M};$$
5) if $m(x) \in \mathfrak{M}$, then $|x|\, m(x) \in \mathfrak{M}$.

Let a class of estimates $\mathfrak{M}$ be given. We shall say that a sequence of functions $F_n(x)$ $(n = 1, 2, ...)$ $\mathfrak{M}$-almost uniformly converges to a function $F(x)$, and write $F_n(x) \to F(x)$, if this sequence converges to $F(x)$ uniformly on every finite interval and, moreover,

$$\sup_n |F_n(x)| \in \mathfrak{M}.$$

A sequence f of functions $f = \{f_n(x)\}$ is called $\mathfrak{M}$-fundamental if there exists an integer $k \geq 0$ and an $\mathfrak{M}$-almost uniformly convergent sequence of k-times continuously differentiable functions $F = \{F_n(x)\}$, for which

$$F_n^{(k)}(x) = f_n(x) \quad (n = 1, 2, ...)$$

(or, in shorter notation, $F^{(k)} = f$).

We shall say that two $\mathfrak{M}$-fundamental sequences

$$f = \{f_n(x)\} \quad \text{and} \quad \varphi = \{\varphi_n(x)\}$$

are $\mathfrak{M}$-equivalent if there exists an integer $k \geq 0$ and sequences $F = \{F_n(x)\}$ and $\Phi = \{\Phi_n(x)\}$, for which

$$F^{(k)} = f, \qquad \Phi^{(k)} = \varphi, \qquad F_n(x) - \Phi_n(x) \to 0.$$

The introduced notion of $\mathfrak{M}$-equivalence is symmetric, reflexive, and transitive; hence the totality of all $\mathfrak{M}$-fundamental sequences decomposes into classes of $\mathfrak{M}$-equivalence, each of which, by definition, is an $\mathfrak{M}$-generalized function in the sense of Mikusiński—Korevaar. We denote the set of these generalized functions (the symbol $\mathfrak{M}$ will be omitted) by $M_{\mathfrak{M}}$. This definition reduces to the definition of Mikusiński—Korevaar if $\mathfrak{M}$ is the collection of all nonnegative functions.

If $\bar{f} \in M_{\mathfrak{M}}$, $\bar{\varphi} \in M_{\mathfrak{M}}$, then by $\alpha\bar{f} + \beta\bar{\varphi}$, by definition, is meant the generalized function containing $\alpha f + \beta\varphi$ ($f \in \bar{f}$, $\varphi \in \bar{\varphi}$). Thus the set $M_{\mathfrak{M}}$ becomes a linear space.

Every generalized function $\bar{f} \in M_{\mathfrak{M}}$ contains a sequence ω consisting of infinitely differentiable functions; by definition, the derivative $\bar{f}^{(l)}$ ($l = 1, 2, \ldots$) is understood to be the generalized function containing $\omega^{(l)}$.

If $f(x)$ is a function for which the sequence $\{f(x), f(x), \ldots\}$ is fundamental (this will be so if and only if there exists an integer $k \geq 0$ and a k-times continuously differentiable function $F(x)$ for which $|F(x)| \in \mathfrak{M}$, $F^{(k)}(x) = f(x)$), then by $\hat{f}$ is meant the generalized function containing this sequence. Thus a one-to-one mapping is defined of the linear space $C_{\mathfrak{M}}$ of such functions $f(x)$ onto the subspace $\hat{C}_{\mathfrak{M}}$ of the space $M_{\mathfrak{M}}$.

Let us note some properties of generalized functions. $\hat{f}^{(l)} = \hat{\varphi}$ if and only if the function $f(x)$ is l times continuously differentiable and $f^{(l)}(x) = \varphi(x)$. If $\bar{f}' = \hat{F}$, then

$$\bar{f} = \int_0^x \widehat{F(t)\,dt} + \hat{C},$$

where C is some constant.

Every generalized function can be represented as a derivative of some order of a generalized function from $\hat{C}_{\mathfrak{M}}$ ("Schwartz' s theorem").

We introduce the definition of (strong) convergence in the space $M_{\mathfrak{M}}$ in the same way, following Mikusiński. If $\bar{f}_n \in M_{\mathfrak{M}}$ ($n = 1, 2, \ldots$) and $\bar{f} \in M_{\mathfrak{M}}$, then we shall write that $\bar{f}_n \to \bar{f}$ if, for some integer $k \geq 0$, there exist functions $F_n(x)$ ($n = 1, 2, \ldots$) and $F(x)$ for which $F_n(x) \to F(x)$ in $\mathfrak{M}$, $\hat{F}_n^{(k)} = \bar{f}_n$ ($n = 1, 2, \ldots$),

$\hat{F}^{(k)} = \bar{f}$. If $\bar{f}_n \to \bar{f}$, then $\bar{f}'_n \to \bar{f}'$. The subspace $\hat{C}_{\mathfrak{M}}$ is everywhere dense in $M_{\mathfrak{M}}$.

We now introduce the collection $O_{\mathfrak{M}}$ of "$\mathfrak{M}$-basic functions" $s(x)$, infinitely differentiable and such that

$$s^{(k)}(x)m(x) \to 0 \quad (k = 0, 1, ...)$$

as $|x| \to \infty$ for every $m(x) \in \mathfrak{M}$. It is obvious that $O_{\mathfrak{M}}$ is a linear space (with respect to the usual operations).

If $\bar{f} \in M_{\mathfrak{M}}$, $\bar{f} = \hat{F}^{(k)}$, $|F(x)| \in \mathfrak{M}$, and $s(x) \in O_{\mathfrak{M}}$, then the integral

$$(s, \bar{f}) = (-1)^k \int_{-\infty}^{\infty} s^{(k)}(x) F(x)\, dx$$

converges and depends only on $\bar{f}$ and $s(x)$, and if $\bar{f}_n \to \bar{f}$, then $(s, \bar{f}_n) \to (s, \bar{f})$.

Define convergence in the space $O_{\mathfrak{M}}$ by saying that $s_n(x) \to s(x)$ ($s_n(x) \in O_{\mathfrak{M}}$; $n = 1, 2, \ldots$; $s(x) \in O_{\mathfrak{M}}$) if, for every $k = 0, 1, ...$, the sequence $s_n^{(k)}(x)$ converges to $s^{(k)}(x)$ uniformly and if

$$m(x) \sup_n |s_n^{(k)}(x)| \to 0$$

as $|x| \to \infty$ for every estimate $m(x) \in \mathfrak{M}$. (In the case where $\mathfrak{M}$ consists of all nonnegative functions, the space $O_{\mathfrak{M}}$ coincides with Schwartz' s space of finite functions; Schwartz ([3]) and I. M. Gel' fand and G. E. Shilov ([4]) also considered some other concrete spaces of basic functions.)

The notion of convergence introduced is weak convergence with respect to the scalar product $(s, \bar{f})$, as is stated by the following theorem.

Theorem 1. *Let $s_n(x) \in O_{\mathfrak{M}}$ $(n = 1, 2, ...)$, $s(x) \in O_{\mathfrak{M}}$. In order that, for every generalized function $\bar{f} \in M_{\mathfrak{M}}$,*

$$(s_n, \bar{f}) \to (s, \bar{f}),$$

it is necessary and sufficient that

$$s_n(x) \xrightarrow{\mathfrak{M}} s(x).$$

The scalar product $(s, \bar{f})$ makes it possible to introduce in the space $M_{\mathfrak{M}}$, besides strong convergence, also weak convergence: namely, for $\bar{f}_n \in M_{\mathfrak{M}}$ $(n = 1, 2, ...)$,

$\bar{f} \in M_{\mathfrak{M}}$, we shall say that $\bar{f}_n \xrightarrow{\text{sl}} \bar{f}$ if $(s, \bar{f}_n) \to (s, \bar{f})$ for every function $s(x) \in O_{\mathfrak{M}}$. If $\bar{f}_n \xrightarrow{\text{sl}} \bar{f}$, then $\bar{f}'_n \xrightarrow{\text{sl}} \bar{f}'$. We note that the weak limit is unique.

The scalar product $(s, \bar{f})$ is a linear continuous functional. It turns out that there are no other linear continuous functionals in the space $M_{\mathfrak{M}}$.

Theorem 2. *Every linear functional $k(\bar{f})$ $(\bar{f} \in M_{\mathfrak{M}})$, continuous with respect to strong convergence, can be represented in the form $(s, \bar{f})$ for some function $s(x) \in O_{\mathfrak{M}}$, which is determined uniquely by the formula*

$$s(a) = \frac{1}{2} k\Big(\widehat{(x-a) + |x-a|''}\Big) \quad (-\infty < a < \infty).$$

This theorem establishes a natural one-to-one correspondence between the space of linear continuous functionals on the space $M_{\mathfrak{M}}$ and the space of basic functions $O_{\mathfrak{M}}$. It is an isomorphism with respect to linear operations and, if the derivative of a functional $k(\bar{f})$ is defined by the formula $k'(\bar{f}) = -k(\bar{f}')$, also with respect to differentiation. If weak convergence is introduced in the space of linear continuous functionals on $M_{\mathfrak{M}}$, then the indicated correspondence will preserve convergence.

By the Schwartz space $S_{\mathfrak{M}}$ we shall mean the collection of linear continuous functionals on the space $O_{\mathfrak{M}}$, this collection being endowed with the usual linear operations, differentiation according to the formula $T'(s) = -T(s')$ $(s(x) \in O_{\mathfrak{M}})$, and weak convergence.

Theorem 1 shows that, for fixed $\bar{f} \in M_{\mathfrak{M}}$, the scalar product $(s, \bar{f})$ is a linear continuous functional on the space $O_{\mathfrak{M}}$, i.e., an element of the space $S_{\mathfrak{M}}$, which we shall denote by $\bar{f}^*$, so that $\bar{f}^*(s) = (s, \bar{f})$. This relation defines a natural mapping of the space $M_{\mathfrak{M}}$ onto a subspace $M^*_{\mathfrak{M}}$ of the space $S_{\mathfrak{M}}$. This mapping is one-to-one and is an isomorphism with respect to linear operations and the operation of differentiation. Moreover, this mapping is continuous, since the convergence $\bar{f}^*_n \to \bar{f}^*$ is equivalent to the convergence $\bar{f}_n \xrightarrow{\text{sl}} \bar{f}$. The space $M^*_{\mathfrak{M}}$ consists of all functionals of finite order, i.e., functionals of the form $T^{(k)}(s)$ $(k = 0, 1, ...)$, where

$$T(s) = \int_{-\infty}^{\infty} F(x) s(x)\, dx \quad (|F(x)| \in \mathfrak{M}).$$

We shall say that a class of estimates $\mathfrak{M}$ has a countable basis if there exists a sequence $m_i(x) \in \mathfrak{M}$ $(i = 1, 2, ...)$ such that any estimate $m(x) \in \mathfrak{M}$ does not exceed $m_i(x)$ for all $x \in (-\infty, \infty)$ for some $i = 1, 2, ...$

Theorem 3. If $\mathfrak{M}$ has a countable basis, then $M^*_{\mathfrak{M}} = S_{\mathfrak{M}}$.

Thus, when the class of estimates has a countable basis, each of the spaces $M_{\mathfrak{M}}$ and $O_{\mathfrak{M}}$ is isomorphic to the space of linear discontinuous functionals on the other.

In the case where there is a countable basis, by introducing a norm one can turn the space $O_{\mathfrak{M}}$ into a countably normed linear space. (In this case the spaces $O_{\mathfrak{M}}$ and $S_{\mathfrak{M}}$ become a special case of a broader class of spaces studied by I. M. Gelfand and G. E. Shilov (unpublished).) Using Theorem 3, one could obtain Theorems 1 and 2 by means of the general theory of countably normed linear spaces.

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Note: Figure translations are in progress. See original paper for figures.