ON \(J\)-NONEXPANSIVE OPERATOR FUNCTIONS

1957

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Abstract

Full Text

MATHEMATICS

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ON J-NONEXPANSIVE OPERATOR FUNCTIONS

(Presented by Academician S. L. Sobolev on 31 V 1957)

1. Let L_+ and L_- be mutually orthogonal complementary subspaces of a Hilbert space H. One of these subspaces, in particular, may be zero. Consider the operator $J = E_+ - E_-$, where E_+ is the projector onto L_+, and E_- is the projector onto L_-. If (f, g) is the scalar product in H, then we introduce in H a nondegenerate indefinite metric by means of the "scalar product" $[f, g] = (Jf, g)$.

We shall consider linear bounded operators in H. An operator $\mathfrak{U}$ will be called J-unitary if it has an inverse in H and, for any f, g from H, $[\mathfrak{U}f, \mathfrak{U}g] = [f, g]$ (which is equivalent to the equality $\mathfrak{U}^*J\mathfrak{U} = J$). An operator Y will be called J-nonexpansive if $[Yf, Yf] \leq [f, f]$. We shall say that an operator Y is two-sided J-nonexpansive if Y and Y^* are nonexpansive (which is equivalent to the simultaneous fulfillment of the inequalities $Y^*JY \leq J$, $YJY^* \leq J$). There exist J-nonexpansive operators that are not two-sided J-nonexpansive. It can be shown that, in order that a J-nonexpansive operator Y be two-sided J-nonexpansive, it is sufficient that one of the following three conditions be fulfilled: 1) $\dim L_- < \infty$, in particular $\dim H < \infty$; 2) $\dim L_+ < \infty$ and the existence of Y^{-1}; 3) Y is an operator of Fredholm type, i.e. $Y = I - T$, where T is a completely continuous operator.

For what follows the following two theorems are of essential importance:

Theorem 1. *The transformation*

$$X = (E_+Y - E_-)(E_+ - E_-Y)^{-1} \tag{1}$$

establishes a one-to-one correspondence between the set of all two-sided J-nonexpansive operators Y and a certain subset of the set of nonexpansive operators X, $\|X\| \leq 1$.

It follows from this that every two-sided J-nonexpansive operator Y admits the representation

$$Y = (E_+X - E_-)(E_+ - E_-X)^{-1}, \tag{2}$$

where $\|X\| \leq 1$.

Theorem 2. *In order that a two-sided J-nonexpansive operator Y admit the polar representation*

$$Y = \mathfrak{U} R, \tag{3}$$

where $\mathfrak{U}$ is J-unitary and the "modulus" R is an operator with nonnegative spectrum such that JR is self-adjoint, it is sufficient that one of the following two conditions be fulfilled: 1) the existence of Y^{-1};

2) Y is an operator of Fredholm type. In this case R is determined uniquely by Y. In case 1), $\mathfrak{U}$ was also determined uniquely.

3) We shall say that the operator-function $Y(\zeta)$ **belongs to the class** $(\mathfrak{R}_J)$ if: a) $Y(\zeta)$ is holomorphic inside the unit disk with the exception, possibly, of a countable set of points; b) there exists a point ζ_0, $|\zeta_0| < 1$, such that $Y^{-1}(\zeta_0)$ exists and $J - Y^*(\zeta_0) J Y(\zeta_0)$ is a completely continuous operator; c) $Y(\zeta)$ is a two-sided J-nonexpansive operator if ζ is a point of holomorphy of $Y(\zeta)$.

The importance of operator-functions of the class $(\mathfrak{R}_J)$ for the study of non-Hermitian and nonunitary operators was clarified by M. S. Livshits ([1,2]) and M. S. Brodskii ([3]). V. P. Potapov ([4]) carried out a detailed investigation of functions of this class for the case $\dim H < \infty$.

Following, in general outline, the path of V. P. Potapov, we have been able to obtain some results for the case of infinite-dimensional H.

3. With the aid of the representations (2) and (3) it is proved that $Y(\zeta) \in (\mathfrak{R}_J)$ has the form

$$Y(\zeta) = \mathfrak{U} \widetilde{Y}(\zeta),$$

where $\mathfrak{U}$ is a constant J-unitary operator, and $\widetilde{Y}(\zeta)$ is an operator of Fredholm type at each point of holomorphy of $Y(\zeta)$. If one now subjects $\widetilde{Y}(\zeta)$ to the transformation (1) and studies the resulting function $\widetilde{X}(\zeta)$, then Theorem 3 can be established.

Theorem 3. *If $Y(\zeta) \in (\mathfrak{R}_J)$, then $Y(\zeta)$, $Y^{-1}(\zeta)$ are holomorphic inside the unit disk, with the exception, possibly, of an isolated set inside the unit disk, at the points of which $Y(\zeta)$, $Y^{-1}(\zeta)$ have poles. Moreover, the leading coefficient of the Laurent series in a neighborhood of a pole is a finite-dimensional operator.*

Introduce elementary factors of the first kind

$$b^{(\mathrm{I})}(\zeta) = \mathfrak{U}^{-1} \left(\frac{\zeta_0' - \zeta}{1 - \bar{\zeta}_0' \zeta} \, |\zeta_0'| \, P + P^{\perp} \right) \mathfrak{U}$$

and of the second kind

$$b^{(\mathrm{II})}(\zeta) = \mathfrak{B}^{-1}\left(\frac{1-\bar{\zeta}_0''\zeta}{\zeta_0''-\zeta}\,|\zeta_0''|\,Q + Q^{\perp}\right)\mathfrak{B},$$

where $|\zeta_0'| < 1,\ |\zeta_0''| < 1$; $\mathfrak{U}$ and $\mathfrak{B}$ are J-unitary operators; P and Q are finite-dimensional projectors such that $P \le E_+$, $Q \le E_-$; $P^{\perp}$ and $Q^{\perp}$ are orthogonal complements.

If ζ_0'' is a pole of $Y(\zeta)$ of multiplicity m, then one can choose $\mathfrak{B}$ and Q so that $Y_1(\zeta) = Y(\zeta)b^{(\mathrm{II})-1}(\zeta)$ has at ζ_0'' a pole of multiplicity less than m, and in addition $Y_1(\zeta) \in (\mathfrak{R}_J)$. An analogous proposition is valid for the poles ζ_0' of the function $Y^{-1}(\zeta)$. The proof is based on passing from $Y(\zeta)$, by means of (1), to the function $X(\zeta)$, $\|X(\zeta)\| \le 1$, and on applying to $X(\zeta)$ a generalization of Schwarz' s lemma, obtained by V. P. Potapov for finite-dimensional matrix-functions and valid for the class of operators under consideration.

4. Define the class $(\mathfrak{R}_J^s)$, consisting of those operator-functions $Y(\zeta) \in (\mathfrak{R}_J)$ for each of which there exists a point ζ_0 in the unit disk such that

$$\mathrm{sp}\{J - Y^*(\zeta_0)JY(\zeta_0)\} < \infty.$$

Theorem 4. *If $Y(\zeta) \in (\mathfrak{R}_J^s)$, then the infinite products ("Blaschke products")*

$$\mathfrak{B}^{(\mathrm{I})}(\zeta) = \prod_{k=1}^{\infty} b_k^{(\mathrm{I})}(\zeta), \qquad \mathfrak{B}^{(\mathrm{II})}(\zeta) = \prod_{k=1}^{\infty} b_k^{(\mathrm{II})}(\zeta), \tag{4}$$

constructed respectively from the poles of $Y^{-1}(\zeta)$ and $Y(\zeta)$, converge in norm uniformly in ζ inside the domain of holomorphy of $Y(\zeta)$. Moreover

$$Y(\zeta) = Y_0(\zeta)\mathfrak{B}^{(I)}(\zeta)\mathfrak{B}^{(II)}(\zeta), \tag{5}$$

where $Y_0(\zeta)$ is an operator-function of the class $(\mathfrak{K}_J^s)$, holomorphic together with $Y_0^{-1}(\zeta)$ inside the unit disk.

The structure of the operator-function $Y_0(\zeta)$ is revealed by Theorem 5.

Theorem 5. The operator-function $Y_0(\zeta) \in (\mathfrak{K}_J^s)$, holomorphic together with $Y_0^{-1}(\zeta)$ inside the unit disk, admits the representation

$$Y_0(\zeta) = \mathfrak{A}_0 \int_0^a \exp\left\{-\frac{e^{i\vartheta(t)}+\zeta}{e^{i\vartheta(t)}-\zeta}\,dE(t)\right\}. \tag{6}$$

Here $\mathfrak{A}_0$ is a J-unitary operator; $\vartheta(t)$ is a monotonically decreasing function ($0 \le \vartheta(t) \le 2\pi$); $JE(t)$ is a Hermitian increasing operator-function ($t = \mathrm{sp}\,JE(t)$); $\int$ is a multiplicative integral.

Theorems 4 and 5 generalize the main theorem of V. P. Potapov ([4]). The essential difficulty that arose in extending this theorem to the infinite-dimensional case consisted in the necessity of a special proof of compactness of the families of operator-functions under consideration. In doing this, we had to use the following criterion.

Compactness criterion. Let the sequences of functionals $\Phi_n^{(1)}(f) \geq 0$, $\Phi_n^{(2)}(f) \geq 0$ converge uniformly on every bounded set H to functionals $\Phi^{(1)}(f)$, $\Phi^{(2)}(f)$ possessing the following properties:

1°. If the sequence $f_i \xrightarrow{\text{sl}} 0$, then $\Phi^1(f_i) \to 0$, $\Phi^{(2)}(f_i) \to 0$.

2°. For every $\varepsilon > 0$ there exists such a finite set Σ_ε that, whatever f in the unit sphere may be, there are $g_\varepsilon^{(1)}, g_\varepsilon^{(2)}$ in Σ_ε such that

$$\Phi^{(1)}(f - g_\varepsilon^{(1)}) < \varepsilon, \qquad \Phi^{(2)}(f - g_\varepsilon^{(2)}) < \varepsilon.$$

If a family of completely continuous operators $\{T_n\}$ is such that for every $f \in H$

$$\|T_n f\| \leq \Phi_n^{(1)}(f), \qquad \|T_n^* f\| \leq \Phi_n^{(2)}(f),$$

then from $\{T_n\}$ one can select a sequence converging in norm.

The stated proposition generalizes the compactness criterion due to M. S. Livshits ([2]).

Let us note that from our results one obtains the theorem of M. S. Livshits on the multiplicative decomposition of the characteristic matrix-function of an operator of class $(i\Omega)$ ([2]). For this it is enough to assume that $Y(\zeta)$ is holomorphic and J-unitary on some arc of the unit circle, to renormalize the elementary factors in (4) and the integrating function in (6), and then to pass from the unit disk to the upper half-plane.

Theorems 4 and 5 were used by V. T. Polyatskii to construct a "triangular model" of an operator T possessing the property that $\operatorname{sp}|I - T^*T| < \infty$ ([5]). They may also find application in the study of certain classes of unbounded operators.

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Received
20 X 1956

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[4] V. P. Potapov, *Tr. Mosk. matem. obshch.*, **4**, 125 (1955).
[5] V. T. Polyatskii. *Tr. 3rd All-Union Math. Congress.* 1. 1956, p. 119.

Note: Figure translations are in progress. See original paper for figures.